simplicial approximation

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The simplicial complex is an algebraic foundation of broader area of mathematics that carries combinatorial information based on the topology. In particular, the well-developed methods of construction and their calculation are significant to apply in real-life data science.

To analyze the topology of data by means of some well known construction such as Čech's, we want an arbitrary map between spaces under some reasonable constraints to well behave around some representing point of data, in a sense that the map does not disregard too much of the combinatorics.

Let $f : |K_1| \to |K_2|$ be a continuous map between realizations of simplicial complexes and $\phi : K_1 \to K_2$ a simplicial map. ϕ is said to be a simplicial approximation of f if $f(\alpha) \in \langle s_2 \rangle \implies |\phi|(\alpha) \in |s_2|$ (equivalently $f(\alpha) \in |s_2| \implies |\phi|(\alpha) \in |s_2|$). At the first sight, this is quite a formal description that the objective is not apparent. When exists, the key properties of such approximation ϕ to f is that it should suffice:

1. homotopic to f,
2. both f and ϕ send each point into the same simplex,
3. the difference is small enough to respect subcomplex $K_1' \subset K_1$ over which f and ϕ coincides.

Keeping these in mind, consider the equivalent statement:

1. $f(\alpha) \in \langle s_2 \rangle \implies |\phi|(\alpha) \in |s_2|$,
2. $f(\alpha) \in |s_2| \implies |\phi|(\alpha) \in |s_2|$.

The later statement automatically implies the former (the former is under more strict condition). From the former to later, we should not mistakenly appeal to the simple topological fact that for an open set U and closed set A with $U \subset A$, $\overline{U} \subset A$; because it is not true that $\overline{f^{-1}(\langle s_2 \rangle)} = f^{-1}(\overline{\langle s_2 \rangle})$ in general (only when f is local homeomorphism) despite that the former statement can be restated as $f^{-1}(\langle s_2 \rangle) \subset |\phi|^{-1}(|s_2|)$.

For the rigorous proof of the equivalence, we note that realized simplices $|s_2| \subset |K_2|$ always are compact metric space, which is sequentially compact (i.e. every sequence has convergent subsequence), which is complete (i.e. every Cauchy sequence converges within the set). Assuming the former statement implies negation of later, in which we have $|\phi|(\alpha) \notin |s_2|$ for some $\alpha \in f^{-1}(\partial |s_2|)$. Choose a convergent sequence $\{\alpha_j\} \subset f^{-1}(\langle s_2 \rangle)$ such that $\beta = \lim \alpha_j \in f^{-1}(\partial |s_2|)$ and an open neighbourhood U of $|\phi|(\beta)$ with property that $U \cap |s_2| = \varnothing$. Then, by continuity of ϕ, for infinitely many j, $|\phi|(\alpha_j) \in U$. Therefore $|\phi|(\alpha_j) \notin |s_2|$ for some (infinitely many) j, contradiction.

To see the approximation as an "operation" to a continuous function between simplicial complexes, there are some notable properties:

1. simplicial approximation is idempotent (simplicial approximation to a given simplicial map f is equivalent to f),
2. simplicial approximation is compatible with composition (of maps);

finally, we have the notable simplicial approximation theorem

Theorem 1. a map $f : (K, K') \to (L, L')$ between simplicial pairs admits simplicial approximation. In particular if K is finite, the approximation can be taken over iterated (barycentric) subdivision of K.

Applying the previous theorem, we have some fundamental results in the homotopy theory regarding the homotopy group of a hypersphere. The proof is from Hatcher's and it uses the key fact that the simplicial map sends a simplex to a simplex of dimension equal to or less than (called degenerate simplex) the original.

Proposition 1. S^n is (n-1)-connected for $n \geq 1$.

Proof. For $m\langle n$, it is sufficient to show that any map $S^m \to S^n$ is null-homotopic. Let s_1, s_2 be (m+1) and (n+1) simplices, respectively. We have $S^m \cong |\dot{s_1}|$ and $S^n \cong |\dot{s_2}|$ and any map $f : S^m \to S^n$ admits simplicial approximation $\phi : Sd^i(\dot{s_1}) \to \dot{s_2}$. Because $dim[Sd^i(\dot{s_1})] = m$, ϕ maps $Sd^i(\dot{s_1})$ to m-skeleton of $\dot{s_2}$. Hence ϕ factors into $Sd^i(\dot{s_1}) \to \dot{s_2} - x \hookrightarrow \dot{s_2}$ for some point x, which is null-homotopic since $\dot{s_2} - x$ is contractible. □

Proposition 2. For $n\rangle 1$, any map $S^n \to S^1$ is null-homotopic.

Proof. By the previous theorem, $\pi_1(S^n) = 0$. Since S^1 is path-connected and locally path-connected, a map $S^n \to S^1$ factors through $S^n \to \mathbb{R}$ by the lifting theorem. □