Limit and Colimit

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I often forget the conceptual background of these well known universal constructions. This bothers me especially when I need to give a new mathematical instrument the conceptual meaning as the extended or restricted version of limit or colimit. Here I write of limit/colimit that include definitions and some concrete examples of instantiating the concept, in order for reminding me of least rigoristic idea that activate the lost memory.

Definition. Let C, D, E are categories and functors F and G are given as: $D \xrightarrow{F} C \xleftarrow{G} E$. The comma category $(F \downarrow G)$ is a category composed of a triple $(d, e, f : Fd \to Ge)$ as an object and the morphism $H : (d, e, f) \to (d', e', f')$ is a pair of morphisms $H_D : d \to d'$ and $H_E : e \to e'$ (we write these only as H) that commutes the following diagram:

$$\begin{array}{ccc} Fd & \xrightarrow{f} & Ge \\ \downarrow{\scriptstyle FH} & & \downarrow{\scriptstyle GH} \\ Fd' & \xrightarrow{f'} & Ge' \end{array}$$

Note that this definition generalizes the case when a functor is an object c of the category C, since c can be regarded as the functor $c : \mathbf{1} \to C$. In this sense, when both the functors F, G are objects $d, e \in C$, then the corresponding comma category is exactly the hom-set written as:

$$(F \downarrow G) = (d \downarrow e) = \hom_C(d, e).$$

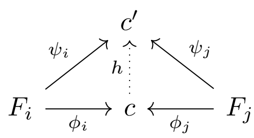
Definition. Let C, J be categories (J is called index category, usually taken to be small or finite). The diagonal functor $\Delta : C \to C^J$ is the functor that maps for each object the constant functor Δc, and for each morphism, the morphism function that assigns each index constantly to the given morphism as in the diagram:

$$\begin{array}{ccc} c = \Delta c_i & = & \Delta c_j = c \\ \downarrow{\scriptstyle f = \Delta f_i} & & \downarrow{\scriptstyle f = \Delta f_j} \\ c' = \Delta c'_i & = & \Delta c'_j = c' \end{array}$$

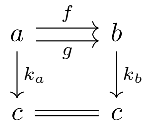
Definition. Let $F : J \to C$ be a functor. If exists, the initial object of comma category $(F \downarrow \Delta)$, regarding the functors F, Δ as to fit in the sequence $\mathbf{1} \to C^J \xleftarrow{\Delta} C$, is called the colimit (of F), denoted $\varinjlim F$.

Definition. Let $F : J \to C$ be a functor. If exists, the terminal object of comma category $(\Delta \downarrow F)$, regarding the functors Δ, F as to fit in the sequence $C \xrightarrow{\Delta} C^J \leftarrow \mathbf{1}$, is called the limit (of F), denoted $\varprojlim F$.

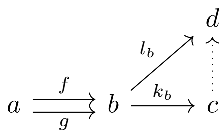
Example. Coproduct is the colimit of $\{F_j\}_{j \in J} \subset Ob(C)$, regarding the corresponding functor $F : J \to C$ as indexed objects. We usually denote $\bigsqcup c_j$ for the coproduct of given indexed objects. In the following diagram, the colimit as the universal arrow is (c, ϕ):



Example. Let J be a category of two objects a, b and two morphisms $f, g : a \to b$. Then an object of the category C^J is exactly a pair of morphisms, also denote $f, g : a \to b$ in C. We call the colimit of $(f, g) \in C^J$ the coequalizer (of (f, g)). An object $(c, k : (f, g) \to \Delta c)$ of comma category $((f, g), \Delta)$ can be represented by a commutative diagram:

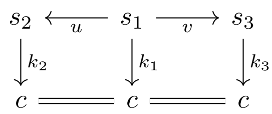


Hence the coequalizer of (f, g) is the pair (c, k) such that for any (d, l), there is unique morphism $c \to d$ in C that commutes the following diagram:



Example. When C is the category of abelian group Ab or the category of R-module RMod for a certain ring R, and suppose $g = 0$, then the coequalizer of (f, g) is the quotient $(b/\mathrm{Im} f, \pi)$.

Example. Let J denotes an index category $\cdot \leftarrow \cdot \rightarrow \cdot$, then an object of C^J is again of the same shape, namely $F = (s_2 \overset{u}{\leftarrow} s_1 \overset{v}{\rightarrow} s_3) \in C^J$. Any element $(c, k : F \to \Delta c) \in C^J$ suffices the commutative diagram:



hence the colimit of F, called pushout of $\mathbf{F}$, is a pair $(c, k : F \to \Delta c)$ that for any pair (d, l), there exists unique morphism $c \to d$ such that the following diagram commutes: