Defining ”wiper fibration”

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Working with a new set of technologies, it sometimes happen in real world that we may better employ an essentially conceptual portion of a mathematical theory, in order to solve practical task in a business.

Today we introduce a type of (Grothendieck) fibration equipped with G-action that we named ”wiper fibration”, obtained out of the theoretical line, despite that it first came across when we were brain storming over the neural network model.

Definition. Let \( M \) be a real n-manifold and \( PT(M) \) its associated projective bundle of the tangent bundle, where the fiber is homeomorphic to \( \mathbb{R}P^{n-1} \). Suppose there given a (not necessarily continuous) function \( \mu : M \times \mathbb{R}P^{n-1} \to [0, \infty) \) of finite value and a topological group \( G \) that acts freely and transitively on \( \mathbb{R}P^{n-1} \). The subspace \( E \) of \( PT(M) \) is given by restricting fibers as:

\[
E = \{(x, v) \in M \times \mathbb{R}P^{n-1} : v = \sup \mu(x, \mathbb{R}P^{n-1})\},
\]

and we call the data \((E, M, G, \mu)\) together with the canonical projection \( E \to M \), the wiper fibration.

Proposition 1. wiper fibration \((E, M, G, \mu)\) is a Grothendieck fibration on a discrete category of sets if for any \((x, v) \in E\) and \(y \in B\), there exists the unique \(g \in G\) such that \(gv \in p^{-1}(y)\).

Proof. For any \((x, v) \in E\) and a function \(f\) such that \(f(y) = x\), we have the unique function \(\phi : E \to E\) such that \(\phi(y, v) = (f(y), gv, v)\), where \(gv \in G\) can be uniquely chosen by assumption. \(\square\)

Restricting attention to a compact subset \(B\) of \(\mathbb{R}^2\), it may be easily imagined that the corresponding wiper fibration well depicts the coordinate and direction of windscreen wiper on which the maximum amount of rain drops are loaded.

For a more practical use, the \(G\) action and the condition of \(\mu\) should be refined in the following way. The choice of \(g \in G\) may better to require a sense of continuity so it still remains fibration over a topological category. The function \(\mu\) wants to be consistent so the fiber behaves well.