Boolean algebra v.s. Heyting algebra

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1. Boolean algebra $(\wedge, \vee, \neg)$: complemented distributive lattice,
2. Heyting algebra $(\wedge, \vee, \rightarrow)$: distributive bounded lattice equipped with implication.

Proposition 1. The following implication axioms of Heyting algebra H are equivalent:

- $(a \rightarrow b) = \max\{x \in H : a \wedge x \leq b\}$.
- $(x \wedge a) \leq b \iff x \leq (a \rightarrow b), \quad \forall x \in H$.

Proof. Since H is bounded, the second statement is only the definition of maximum element under the given condition. □

Proposition 2. Boolean algebra is a Heyting algebra.

Proof. Given a boolean algebra B, we can canonically define the implication $a \rightarrow b$ as:

$$a \rightarrow b \iff \neg(a \wedge \neg b),$$

in which the later implies $(\neg a \vee b)$ by the De Morgan's law. Therefore it is enough to check if the implication suffices the axiom:

$$(c \wedge a) \leq b \iff c \leq (a \rightarrow b); \quad a, b, c \in B$$

For the only if part, assume $c \leq (a \wedge \neg b)$, then we have:

$$\begin{aligned}
&(c \leq a) \wedge (c \leq \neg b) \\
\Longrightarrow\ &((c \wedge a) \leq (b \wedge a)) \wedge ((c \wedge a) \leq (\neg b \wedge a)) \\
\Longrightarrow\ &(c \wedge a) \leq ((\neg b \wedge a) \wedge (b \wedge a)) \\
\Longrightarrow\ &(c \wedge a) \leq 0
\end{aligned}$$

The premise $(c \wedge a) = 0$ indicates that the conclusion $c \leq (a \rightarrow b)$ is universal, which is not the case, contradiction.

For the if part, we have $c \leq (\neg a \vee b)$ by assumption. Then,

$$\begin{aligned}
&(c \wedge a) \leq ((\neg a \vee b) \wedge a) \\
\Longrightarrow\ &(c \wedge a) \leq (b \wedge a) \leq b.
\end{aligned}$$

□

Proposition 3. In a Heyting algebra H, by defining pseudo-negation by $\neg a := (a \rightarrow 0)$, the following holds:

$$a \leq \neg\neg a.$$

Proof. For an arbitrary $a \in H$, we have

$$\begin{aligned}
&(a \wedge (a \rightarrow 0)) \leq 0 \\
\iff\ &a \leq ((a \rightarrow 0) \rightarrow 0) = \neg\neg a
\end{aligned}$$

□

Corollary 1.

$$\neg a \wedge a = 0, \forall a \in H$$

Proof.

$$a \leq \neg\neg a = (\neg a \to 0) \iff a \wedge \neg a \leq 0.$$

□

Proposition 4. In a Heyting algebra H, the following conditions are equivalent:

1. the excluded middle: $(\forall a \in H)(a \vee \neg a = 1)$,
2. the double negation elimination: $(\forall a \in H)(\neg\neg a = a)$.

Proof. From 1 to 2,

$$\begin{aligned} &\neg\neg a = (\neg\neg a \wedge a) \\ \iff &\neg\neg a \leq a. \end{aligned}$$

From 2 to 1, suppose $a \wedge \neg a < 1$ for some $a \in H$. Then $a \wedge \neg\neg a < \neg\neg a$ is deduced that shows $a < \neg\neg a$. □

Proposition 5. De Morgan's Law In a Heyting algebra H, the followings holds:

1. $\neg(a \vee b) = \neg a \wedge \neg b, \quad \forall a, b \in H,$
2. $\neg(a \wedge b) = \neg\neg(\neg a \vee \neg b), \quad \forall a, b \in H$

Proof. For 1, because $a \leq a \vee b$, we have:

$$\begin{aligned} &a \wedge \neg(a \vee b) \leq ((a \vee b) \wedge \neg(a \vee b)) = 0 \\ \iff &\neg(a \vee b) \wedge a \leq 0 \\ \iff &\neg(a \vee b) \leq \neg a. \end{aligned}$$

Analogous argument holds for b, hence we have $\neg(a \vee b) \leq (\neg a \wedge \neg b)$.

On the other hands, observe that $(\neg a \wedge b) \wedge (a \vee b) = (\neg a \wedge b) \wedge (a \wedge \neg b) = 0$. Thus we have $(\neg a \wedge \neg b) \leq \neg(a \vee b)$. 1 is proved.

For 2,

$$\begin{aligned} \neg(a \wedge b) \wedge \neg(\neg a \vee \neg b) &= \neg((a \wedge b) \vee (\neg a \vee \neg b)) \\ &= \neg(((a \wedge b) \vee \neg a) \vee ((a \wedge b) \vee \neg b)) \\ &= \neg((\neg a \vee b) \vee (a \vee \neg b)) \\ &= \neg 1 \\ &= 0. \end{aligned}$$

Hence we have $\neg(a \wedge b) \leq \neg\neg(\neg a \vee \neg b)$.

On the other hands, we have

$$\begin{aligned} &\neg\neg(\neg a \vee \neg b) \wedge (a \wedge b) \\ =&\neg(\neg\neg a \wedge \neg\neg b) \wedge (a \wedge b) \\ \leq&\neg(\neg\neg a \wedge \neg\neg b) \wedge (\neg\neg a \wedge \neg\neg b) \\ =&0. \end{aligned}$$

Hence we have $\neg\neg(\neg a \vee \neg b) \leq \neg(a \wedge b)$. The proof completed. □