# Step by step construction of simplicial set

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**Directed graph** is determined by a quadruple $(O, M, dom, cod)$ where O is a set of objects, M is that of morphisms and dom and cod assign to each morphism its domain and codomain, respectively.

$$M \underset{cod}{\overset{dom}{\rightrightarrows}} O$$

Because a morphism between directed graphs is a pair of maps $M \to M'$ and $O \to O'$ that is compatible with dom/cod, we have a category of directed graph **Grph**. Moreover, a directed graph is regarded as a "category" without (requirement of) identity and composition laws. It is well known that for given directed graph, there exists the category upto equivalence that is "generated by" the directed graph in the sense that the category is obtained as the universal arrow $\mathrm{Init}(g \downarrow U)$ for each $g \in \mathbf{Grph}$ where U is the forgetful functor $\mathbf{Cat} \to \mathbf{Graph}$ that maps to each category the underlying graph. These categories are called **free categories** and the process to obtain a free category out of given graph is called **free construction**.

Let $\mathbf{\Delta}$ be **simplex category** whose objects are free categories of inhabited (non-empty) finite linear directed graphs (This description is rigorous and may be somewhat intimidating owing to the categorical term, but it only says that an object has a representation with non-empty finite sequence of ordered integers $\Delta_n = [\sigma(0) \to \sigma(1) \to \ldots \to \sigma(n)]$, where $\sigma$ is order preserving map. We often express an object concisely as $\Delta_n = [0, 1, \ldots, n]$). What is implicit here is that an order preserving map between ordinals $[k] \to [n]$ determines a morphism in the skeletal category.

A **simplicial set** is a contravariant functor $\mathbf{\Delta} \to \mathbf{Set}$, a categorical generalization of so called singular $\Delta$-complex. The construction of singular $\Delta$-complex out of given simplicial set $S$ is as follows.

- For each $n \geq 0$, $S$ assigns to an object $\Delta_n \in \mathbf{\Delta}$ a set of n-simplices $X_n \in \mathbf{Set}$;
- For each order preserving injective map $g : \Delta_k \to \Delta_n \, (k \leq n)$, $S$ assigns $S(g) = g^* : X_n \to X_k$ called *face map*;
- For each order preserving surjective map $g : \Delta_n \to \Delta_k \, (k \leq n)$, $S$ assigns $S(g) = g^* : X_k \to X_n$ called *degeneracy map*;
- Note that any map $g \in \Delta$ can be decomposed into surjection followed by injection in the form of $A \to g(A) \to B$, a functorial property of $S$ attests that $S(g)$ defines maps between simplices of fixed dimensions called *simplicial map*.

Now define a set $X$ as a quotient of disjoint union:

$$X = \bigsqcup_{n=0} (X_n \times \Delta^n)/\sim,$$

where the equivalence is given by the identification of images of maps $1 \times g_* = g^* \times 1$ as in:

$$X_n \times \Delta^n \xleftarrow{1 \times g_*} X_n \times \Delta^k \xrightarrow{g^* \times 1} X_k \times \Delta^n,$$

and $g_* : \Delta^k \to \Delta^n$ is the induced map of $g \in \mathbf{\Delta}$. Note that $\Delta_n$ is regarded as an extracted combinatorial data of $\Delta^n$ apart from categorical association, while $\Delta^n$ describes the categorical generalization in terms of ordered graph hence it implicitly admits composition compatibility along with functors.

On the left, the element represents a pair of an element of $X_n$ and a point in a k-face of n-simplex whereas the right a pair of k-face of an element of $X_n$ and a point in a k-simplex.

Seeing *singular* $\Delta$-complex as a special version of CW complex with its characteristic maps being simplicial, or as an extended version of $\Delta$-complex with the attaching maps are allowed to be degenerate (hence a characteristic map can have the dimension lower than 1 in its boundary image), this interpretation is exactly the description of characteristic map $g_j : \Delta_j^n \to X$ where n-skelton is defined by $X_n = X_{n-1} \sqcup_j \Delta_j^n/(g_j)$ where $g_j$ restricts to the **simplicial** attaching map $g_j|_{\partial\Delta_j^n} : \partial\Delta_j^n \to X_{n-1}$ (rigorously, this needs a proof [A.Hatcher, Proposition A.19]).