K-Theory 1-2

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1 Preliminary

1.1 group and its action

1.1.1 group action in algebraic setting

Definition. G:group, X:set.

$$\triangleright : G \times X \to X; (g, x) \mapsto g \triangleright x.$$

is called the action of G on the left on X (a.k.a. $G \curvearrowright X$) when the following conditions are satisfied:

1. associativity: $(g \cdot h) \triangleright x = g \triangleright (h \triangleright x)$,
2. identity: $e \triangleright x = x$.

Note. For arbitrary $g \in G$, $x \mapsto g \triangleright x$ is a bijection of X.

Note. By the exponential law in set, which asserts that there is an one-to-one relationship between maps from the product and maps of maps, namely:

$$X^{G \times X} \cong (X^X)^G.$$

We can consider the group action as a group homomorphism of G to the symmetric group of X since the functorial property of groups recover the associativity and identity together with one-to-one mapping obtained by exponential law. Hence G is said to act on X if a group homomorphism is given by:

$$\rho : G \to \mathrm{Sym}(X)$$

Note. The right action is also defined by reversing the direction from which the left group action defined. The statement corresponding to *associativity* is then given by:

1. ' $x \triangleleft (h \cdot g) = (x \triangleleft h) \triangleleft g$.

Note. $G \triangleright G$ and $G \triangleleft G$.

1.1.2 induced right action from left action

$G \triangleright X$ (X is G-set). Then,

$$\triangleleft : X \times G \to X; (x, g) \mapsto g^{-1} \triangleright x$$

is a (induced) right action.

Example. juxtaposition of loops as right action.

Example. composition of bijection as left action.

1.1.3 G-equivalent maps

Definition.

$$G \in \mathbf{Grp}, X, Y \in \mathbf{Set}$$
$$G \triangleright X, G \triangleright Y, \quad f : X \to Y \quad map$$

f is G-equivalent iff the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g\triangleright -}\downarrow & & \downarrow{\scriptstyle g\triangleright -} \\ X & \xrightarrow{f} & Y \end{array}$$

1.1.4 type of actions

Definition. (transitive action)

$$G \triangleright X$$

G-action is called **transitive** on X iff

$$(\forall x, y \in X)(\exists g \in G) \text{ s.t. } g \triangleright x = y.$$

Note. Transitive action gives the acting set a sense of symmetry, which is somewhat corresponding to the translation invariance of a vector space, in which the action is defined by sum for each element.

Definition. (faithful action)

$$G \triangleright X$$

G-action is called **faithful** on X iff

$$(\forall g, h \in X \text{ s.t. } g \neq h)(\exists x \in X) \text{ s.t. } gx \neq gy, \text{ or equivalently,}$$
$$\triangleright : G \to \text{Sym}(X) \text{ has trivial kernel.}$$

1.1.5 group action in topological setting

Definition. G: top. group., X: top. space.

$$\triangleright : G \times X \to X; (g, x) \mapsto g \triangleright x.$$

is called the continuous action of G on the left on X if the it suffices the following conditions:

1. $G \curvearrowright X$,
2. $\triangleright$ is continuous on product topology.

Note. A continuous faithful action of topological group G on a space X is thought of as a group of homeomorphism. This is because $g \in G$ defines a continuous bijection $\{g\} \times X \to X$ and so it does its inverse g^{-1}. Faithfulness asserts that $G \subset \text{Homeo}(X)$ as the subgroup.

1.1.6 examples

Example. $\mathbb{Z}_2$ acts (as product group) on any Abelian group X by

$$(\pm 1, x) \mapsto \pm x$$

Example. Symmetry group $\text{Sym}(X)$ of a set X acts on X by

$$(\sigma, x) \mapsto \sigma x$$

Example. $\text{GL}(n, \mathbb{R}) \curvearrowright \mathbb{R}^n$

Example. The rotation in $X = S^1$ can be thought of as $S^1 \curvearrowright X$, by:

$$(S^1, X) \ni (e^{is}, e^{it}) \mapsto e^{i(s+t)} \in X$$

2 Vector Bundle

2.1 Basics

2.1.1 Definition of vector bundle

Definition. A map $p : E \to B$ is called n-dimensional real vector bundle if all the following statements are hold:

1. *fiber isomorphisms*: $\forall x \in B$, there exists an isomorphism $\mathbb{R}^n \to p^{-1}(x)$.

2. *local triviality*: $\forall x \in B, \exists U \in \mathcal{V}(x)$, there exists a homeomorphism $h_U : U \times \mathbb{R}^n \to p^{-1}(U)$ such that $p(h_U(x,v)) = x$ and h_U is restricted to the linear isomorphism $\{x\} \times \mathbb{R}^n \to p^{-1}(x)$. h_U is called *coordinate function.*

3. *coordinate transformations* For a pair of coordinate functions h_U, h_V with $U \cap V \neq \varnothing$, the homeomorphism $g_{VU} = h_V^{-1} \circ h_U : U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n$ restricts to a homeomorphism of $\mathbb{R}^n$ for each $x \in U \cap V$ that coincides with the action of G, namely:

$$g_{VU}(x,v) = (x, gv) \text{ for some } g \in G.$$

g_{VU} is called *coordinate transformation* or *transition function* from the coordinate neighbourhood U to V. By exponential law together with the first two conditions, a transition function is thought of implicitly as a map $g_{VU} : U \cap V \to G$.

Note. From the definition, one can immediately deduce the following properties:

- *group of the bundle*: A subgroup G of $\mathrm{GL}(n,\mathbb{R})$ is given to faithfully and continuously act on $\mathbb{R}^n$ (depending on the local charts). G is called *group of the bundle* or in a literature *structure group.*
- *cocycle condition* 1-Čech cochain of coefficients G is only a map from $U \cap V$ to G, defined when $U \cap V \neq \varnothing$. Hence transition functions are thought of 1-Čech cochains on X. Every transition functions suffices the following condition corresponding to the cocycle condition:

$$g_{WV} \cdot g_{VU} = g_{WU}, \quad \text{if } U \cap V \cap W \neq \varnothing.$$

2.1.2 Isomorphism of vector bundle

Definition. An isomorphism between vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over the same base space B is a homeomorphism $h : E_1 \to E_2$ taking each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism. Thus an isomorphism preserves all the structure of a vector bundle, so isomorphic bundles are often regarded as the same. We use the notation $E_1 \cong E_2$ to indicate that E_1 and E_2 are isomorphic.

Note. A map $h : E_1 \to E_2$ between vector bundles $p_i : E_i \to B_i$ over homeomorphic bases $f : B_1 \to B_2$ is an isomorphism iff:

h takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(f(b))$ by a linear isomorphism,

Note. Assume that an isomorphism h between vector bundles E_1, E_2 are given as:

$$\begin{array}{ccc} E_1 & \xrightarrow{h} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}.$$

Then the coordinate transformation g'_{21} of E_2 from U_1 to U_2 is expressed with g_{21} of E_1 in the form:

$$g'_{21} = h g_{21} h^{-1},$$

where h is appropriately restricted on $U_1 \cap U_2$. This is seen as in the following commutative diagram:

$$\begin{array}{ccc} (U_1 \cap U_2) \times \mathbb{R}^n & \xrightarrow{h} & (U_1 \cap U_2) \times \mathbb{R}^n \\ \downarrow{\scriptstyle g_{21}} & & \downarrow{\scriptstyle g'_{21}} \\ (U_1 \cap U_2) \times \mathbb{R}^n & \xrightarrow{h} & (U_1 \cap U_2) \times \mathbb{R}^n. \end{array}$$

2.1.3 Fundamental examples of vector bundle

Example. *trivial bundle* The canonical projection $p : X \times \mathbb{R}^n \to X$ of product space is a vector bundle called trivial bundle over X. The structure group can be reduced to the trivial group $\{e\} \subset \mathrm{GL}(n,\mathbb{R})$.

Example. *M̈obius bundle* M is defined as a quotient space of $I \times \mathbb{R}$ under the identification of $(0,t) \sim (1,-t)$. The canonical projection $p' : I \times \mathbb{R} \to I$ induces universal arrows (*pushout*) $I \xrightarrow{q} S^1 \xleftarrow{p} M$ that commutes the following diagram:

$$\begin{array}{ccc} I \times \mathbb{R} & \xrightarrow{q'} & M \\ \big\downarrow{\scriptstyle p'} & & \vdots{\scriptstyle p} \\ I & \xrightarrow{q} & S^1. \end{array}$$

The fiber $p^{-1}(x)$ is homeomorphic to $\mathbb{R}$ and the group of bundle can be reduced to $\mathbb{Z}_2 \cong \{\pm 1\} \subset \mathrm{GL}(1,\mathbb{R})$. To justify that $\mathbb{Z}_2$ acts on the model fiber $\mathbb{R}$ of *M̈obius bundle* along with efficiently chosen local trivializations, one will need to identify M with so called *canonical line bundle* $\mathfrak{L}_{\mathbb{R}P^1} \to \mathbb{R}P^1$, we introduce later.

Example. *canonical line bundle* $E = \mathfrak{L}_{\mathbb{R}P^1}$ is defined as:

$$p : E = \{(l,v) \in \mathbb{R}P^1 \times \mathbb{R}^2 : v \in l\} \to \mathbb{R}P^1; \quad p(l,v) = l.$$

Note. The line in $\mathbb{R}P^1$ is considered to be embedded in $\mathbb{R}^2$ as the 1-dimensional subspace without antipodal identification (therefore the $v \in l$ can be zero despite the original line $l \in \mathbb{R}P^{‘}$ is defined on $\mathbb{R}^2 \backslash \{0\}$).

Lemma 1. *$M \cong \mathfrak{L}_{\mathbb{R}P^1}$ as line bundles over the base space $S^1 \cong \mathbb{R}P^1$.*

Proof. Each $s \in [0,1]$ determines the line passing through both the origin and $e^{i\pi s} \in S^1$ in $\mathbb{R}^2$, so let us denote the line l_s. Let ϕ be a map defined by:

$$\phi : M \to \mathfrak{L}_{\mathbb{R}P^1}; \quad \phi([s,t]) = [l_s, t]',$$

where the identification $[,]$ in M is given by $(0,t) \sim (1,-t)$ as before, and $[,]'$ in $\mathfrak{L}_{\mathbb{R}P^1}$ by $(l_s,t) \sim (l_{(-s)}, -t)$. The later identification is deduced from the fact that rotating the line l_s by π degree defines a transposition $(l_s,t) \mapsto (l_{(-s)}, -t)$, which must coincide in $\mathfrak{L}_{\mathbb{R}P^1}$. To check ϕ being isomorphism is a routine. □

Note. It may be confusing at first that (l_s,t) is a different element from $(l_{(-s)}, t)$, as an element of $\mathfrak{L}_{\mathbb{R}P^1}$. This is because the "twistedness" gives rise to the coordinatewise identification may not work as in the ordinal product space $\mathbb{R}P^1 \times \mathbb{R}$.

Example. *tangent bundle of the unit sphere* A vector bundle $p : E \to S^n$ where

$$E = \{(x,v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\}.$$

The local trivialization $h : U_x \times \mathbb{R}^n \to p^{-1}(U_x)$ is defined by $h(y,v) = (y, \pi(v))$, where π is the orthogonal projection to $p^{-1}(y)$.

2.2 Sections

Definition. (*section of a bundle*) A *section* of a given vector bundle $p : E \to B$ is a right inverse map $s : B \to E$ of the projection p, namely such a map with $p \circ s = \mathbb{1}_B$

Note. every vector bundle has canonical section called *zero section*, which assigns zero vector to each point of the base space.

Lemma 2.

$$M \ncong S^1 \times \mathbb{R}$$

Proof.

$$M \backslash M_0 \ncong (S^1 \times \mathbb{R}) \backslash (S^1 \times 0)$$

□

References

[1] N. Steenrod. *The Topology of Fibre Bundles*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1999. ISBN: 9780691005485. URL: https://books.google.co.jp/books?id=m%5C_wrjoweDTgC.