The effect of opposite and extranatural transformation on a cartesian closed category

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op : $\mathbf{Cat} \to \mathbf{Cat}$ is an endfunctor of category of (small) categories that assigns each category C to its categorical dual C^{op} called opposite category and each functor $F : C \to C'$ to the induced functor $F^{\mathrm{op}} : C^{\mathrm{op}} \to C'^{\mathrm{op}}$.

The opposite category has the same objects of original category while its morphisms are reversed in direction. This reversing should be considered carefully because it is only syntactical as well as relative in meaning where the hom-sets have 1 to 1 correspondence $\hom_C(a, b) \cong \hom_{C^{\mathrm{op}}}(b, a)$ (within small sets). Hence $f^{\mathrm{op}} \in C^{\mathrm{op}}(b, a)$ can always be replaced by its dual $f : a \to b$ in C and vice versa.

Then why do we need an opposite?

There may be several reasonable answers to that specific question; nevertheless, we only count on a seemingly essential use that would account for op clarifying a theme of duality.

One such example appears in some categorical notion that is preserved by op, while some not. Similarly when one think of a property that may or may not hold in a category, one may ask what the property looks like, or if they still do or do not hold when the arrows are reversed.

In this direction, we first consider a cartesian closed category. Assuming non-triviality, this type of category fails to stay cartesian closed in the opposite. Furthermore, there is a stronger fact:

Fact. A cocartesian closed category is isomorhpic to trivial category $\mathbf{1}$.

In what follows, note that "product" stands for nothing but a symbol thus it is interpreted according to the context.

Definition. A monoidal category (i.e. equipped with small products, within the reasonable sense) C is called cartesian closed if the following functors admit specified right adjoints:

$$C \to \mathbf{1},\ c \mapsto 0; \quad C \to C \times C,\ c \mapsto \langle c, c\rangle; \quad C \to C,\ a \mapsto a \times b.$$

To the first functor, the right adjoint is $0 \mapsto t$ which is just the specification of the terminal object.

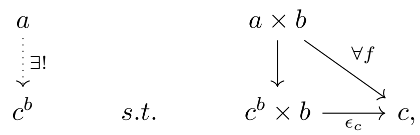
To the second, it is required to have the right adjoint $\langle a, b\rangle \mapsto a \times b$, the assignment of the product of the pair of objects.

To the last, the right adjoint is denoted by $c \mapsto c^b$.

Because C is assumed to have small products, we can deduce that the only non-triviality is for the last adjoint to exist.

Proposition 1. $\mathbf{Set}$ is cartesian closed while $\mathbf{Set}^{\mathrm{op}}$ is not.

Proof. Since there is a bijection between hom-sets $\phi : \hom(a \times b, c) \to \hom(a, \hom(b, c))$, the component of counit $\epsilon_c : c^b \times b \to c$ suffices the universal property:



where $\phi(f)$ is the unique map that concludes $c^b = \hom(b, c)$ in $\mathbf{Set}$. Therefore $\mathbf{Set}$ is cartesian closed.

For $\mathbf{Set}^{\mathrm{op}}$, we can see there are right adjoints for the first two functors of our concern. The terminal object is to the initial object $\varnothing \in \mathbf{Set}$ as the cartesian product is to the disjoint union in the opposite. The last condition can be immediately checked by setting $c = \varnothing$ where there is no way to suffice $\hom(c, a \sqcup b) \cong \hom(c^b, a)$ in $\mathbf{Set}$. $\square$

We introduce further more examples of cartesian closed categories.

Proposition 2. For any set U, the power set $P(U)$ as a preorder is cartesian closed.

Proof. Every morphisms are inclusions hence the terminal object is U. Any diagram $a \leftarrow c \rightarrow b$ in $P(U)$ is equivalent to $c \rightarrow a \cap b$ (unique correspondence) hence $\langle a, b\rangle \mapsto a \cap b$ is the (second) right adjoint functor. Now that it is enough to show that $c \mapsto c^b = c \cup b^{-1}$ is the right adjoint of $a \mapsto a \cap b$.

Over a direct (set theoretical) discussion, we see that there is 1 to 1 correspondence $(a \cap b \leq c) \cong (a \leq c \cup b^{-1})$ along with the universal property of counit $\epsilon_a : a \cap b \rightarrow a$ as in:

$$\begin{array}{ccccc} c & & c \cap b & & \\ \Big\downarrow{\scriptstyle \exists!} & \qquad s.t. \qquad & \Big\downarrow & \searrow^{\forall f} & \\ a \cup b^{-1} & & a \cap b & \xrightarrow[\epsilon_a]{} & a. \end{array}$$

∎

Proposition 3. In a cartesian closed category C, it holds $c \cong c^t, \quad c^{b\times b'} \cong (c^{b'})^b$.

Proof. First we see $b \cong b \times t$ for any $b \in C$ due to the universality as shown in the product diagram:

$$\begin{array}{ccccc} & & b & & \\ & \swarrow^{=} & \Big\downarrow & \searrow & \\ b & \longleftarrow & b \times t & \longrightarrow & t. \end{array}$$

Since C is cartesian closed, we can deduce that the counit gives the isomorphism we are looking for, precisely:

$$\begin{array}{ccccc} c & & c \times t & & \\ \Big\downarrow{\scriptstyle \exists!} & \qquad s.t. \qquad & \Big\downarrow & \searrow & \\ c^t & & c^t \times t & \xrightarrow[\epsilon_c]{} & c, \end{array}$$

when we take the (arbitrary) diagonal map an isomorphism.

By the appropriate choice of objects in the adjoint diagram, we can find the unique morphisms

$$(c^{b'})^b \times t \overset{\exists!}{\dashrightarrow} c^{b\times b'} \overset{\exists!}{\dashrightarrow} (c^{b'})^b$$

that split an isomorphism of our concern.

For rigorousness, we depict a diagram of one direction as follow (the other direction is similar):

$$\begin{array}{ccc} & & (c^{b\times b'})^t \\ & \nearrow & \Big\uparrow \\ (c^{b'})^b & \xrightarrow{\eta_*} & ((c^{b'})^b \times t)^t. \end{array}$$

This implies that $(c^{b'})^b \rightarrow c^{b\times b'}$ is unique if such map exists that the diagram commutes, where the existence is an immediate consequence of one to one correspondence $[a, c^{b\times b'}] \cong [a, (c^{b'})^b]$ for arbitrary $a \in C$ (hence put $a = c^{b\times b'}$ for instance). ∎

Proposition 4. In a cartesian closed category C, what is the natural transformation $b^a \times c^b \rightarrow c^a$ is like? Is it associative?

Proof. When we see the objects as the result of composed functors of $C^{\text{op}} \times C$, the map is considered as the component of natural transformation $\tau : C^{\text{op}} \times C \dot{\rightarrow} C$:

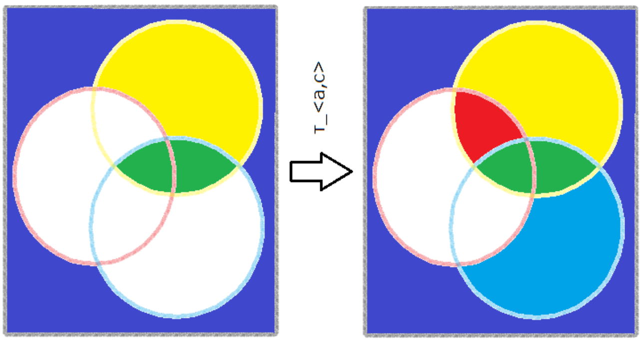
$$\begin{array}{ccccc} C^{\text{op}} \times C & \longrightarrow & \longrightarrow & \longrightarrow & C \\ \Big\downarrow & & & & \Big\uparrow \\ C^{\text{op}2} \times C^2 & \longrightarrow & C^{\text{op}} \times C & \longrightarrow & C \end{array} \qquad \begin{array}{ccccc} \langle a, c\rangle & \longmapsto & \longmapsto & \longmapsto & c^a \\ \Big\downarrow & & & & \Big\uparrow{\scriptstyle \tau_{\langle a,c\rangle}} \\ \langle\langle a, b\rangle, \langle b, c\rangle\rangle & \longmapsto & \langle b^a, c^b\rangle & \longmapsto & b^a \times c^b. \end{array}$$

It is apparent in **Set** that $\tau_{\langle a,c\rangle}$ assigns each composable pair of functions to the composite function through fixed b. When it comes to think of the associativity, we should be careful with the notion of *fixed object* through which τ compose a pair of maps respectively. For this purpose, we denote by $\tau^a_{\langle z,c\rangle}$ as such in the diagram with respect to the associativity:

$$\begin{array}{ccc} a^z \times b^a \times c^b & \xrightarrow{1\times\tau^b_{\langle a,c\rangle}} & a^z \times c^a \\ {\scriptstyle \tau^a_{\langle z,b\rangle}\times 1}\downarrow & & \downarrow{\scriptstyle \tau^a_{\langle z,c\rangle}} \\ b^z \times c^b & \xrightarrow[\tau^b_{\langle z,c\rangle}]{} & c^z. \end{array}$$

For a supplement to the last proposition, the "composition" looks not so obvious in general. We share a drawing that illustrates the case of preorder $P(U)$ below.

Here we assume a pink circle denotes $a \in P(U)$, light blue circle $b \in P(U)$ and yellow circle $c \in P(U)$. The image of $b^a \times c^b$ under $\tau^b_{\langle a,c\rangle}$ is represented by a Venn diagram.



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