Kleisli construction and its properties

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We introduce *Kleisli* construction and its basic properties. Its remarkable arrow composition within the hom-set of *Kleisli* category is translated in the context of computer programs, where the ill-behaviours exhibited by real programs are encoded as the effect of end-functor of associated monad [2]. These ill-behaviours include like nontermination, non-determinism or side-effects.

At the end of the article, we conclude that the *Kleisli* category and its associated adjoints give the universal (initial) object in the category of adjoints over $X \in \mathbf{Cat}$, namely $(X \downarrow \mathbf{Adj})$, whom we don't investigate further for the detail here.

The notions and core ideas are parallel to those of [1] and you may find this article is just a collection of exercise that offer a quick reference to the detailed constructions.

Definition. The *Kleisli category* X_T of a monad $\langle T, \eta, \mu \rangle$ on X is the category whose objects denoted by $x_T \in X_T$ are those of X and the morphisms denoted by $f^\flat : x_T \to y_T$ are every morphisms corresponding to $f : x \to Ty$ in X, together with the specified composition. Hence we have bijections $Ob(X) \cong Ob(X_T)$ and $\hom_X(x, Ty) \cong \hom_{X_T}(x_T, y_T)$.

The specified composition is defined by:

$$g^\flat \circ f^\flat = (\mu_z \circ Tg \circ f)^\flat : x_T \to z_T.$$

Lemma 1. X_T is a category.

Proof. First we see that $(\eta_y)^\flat$ is the identity on y_T, where $\eta_y : y \to Ty$ is the component of monad unit at y. For $f : x \to Ty$, we have $(\eta_y)^\flat \circ f^\flat = ((\mu_y \circ T\eta_y) \circ f)^\flat = f^\flat$ whereas it holds that $g^\flat \circ (\eta_y)^\flat = (\mu_z \circ (Tg \circ \eta_y))^\flat = (\mu_z \circ (\eta_{Tz} \circ g))^\flat = g^\flat$ for $g : y \to Tz$ (i.e. naturality of η).

Now that it is enough to show that $\mu_w \circ Th \circ \mu_z = \mu_w \circ T\mu_w \circ T^2 h$ holds to see the composition is associative due to the following calculation:

$$\begin{aligned} h^\flat \circ (g^\flat \circ f^\flat) &= ((\mu_w \circ Th \circ \mu_z) \circ Tg \circ f)^\flat, \\ (h^\flat \circ g^\flat) \circ f^\flat &= ((\mu_w \circ T\mu_w \circ T^2 h) \circ Tg \circ f)^\flat. \end{aligned}$$

But the equation is canonically induced by the naturality (and the associative law) of $\mu : T^2 \dot{\to} T$ over $h : z \to Tw$ as in:

$$\begin{array}{ccccc} T^2 z & \xrightarrow{\mu_z} & Tz & & \\ {\scriptstyle T^2 h}\downarrow & & \downarrow{\scriptstyle Th} & & \\ T^3 w & \xrightarrow{T\mu_w} & T^2 w & \xrightarrow{\mu_w} & Tw \end{array}$$

□

Lemma 2. Provided with a monad $\langle T, \eta, \mu \rangle$ on X, the *Kleisli* construction $F_T : X \to X_T$ given by $(k : x \to y) \mapsto ((\eta_y \circ k)^\flat : x_T \to y_T)$ is functorial.

Proof. Given a composable morphisms $x \xrightarrow{k} y \xrightarrow{l} z$ on X, we see the following equations hold:

$$\begin{aligned} F_T l \circ F_T k &= (\eta_z \circ l)^\flat \circ (\eta_y \circ k)^\flat \\ &= ((\mu_z \circ T\eta_z) \circ (Tl \circ \eta_y) \circ k)^\flat \\ &= (\mathrm{id}_{Tz} \circ \eta_z \circ l \circ k)^\flat \\ &= F_T(l \circ k). \end{aligned}$$

□

Lemma 3. Provided with a monad $\langle T, \eta, \mu \rangle$ on X, the construction $G_T : X_T \to X$ given by $(f^\flat : x_T \to y_T) \mapsto (\mu_y \circ Tf : Tx \to Ty)$ is functorial.

Proof. It is enough to show that for any composable morphisms $x_T \xrightarrow{f^\flat} y_T \xrightarrow{g^\flat} z_T$ on X_T, the following equation holds:

$$\mu_z \circ T\mu_z \circ T^2 g = \mu_z \circ Tg \circ \mu_y.$$

This is again the canonical consequence of a calculation:

$$\begin{aligned} G_T(g^\flat \circ f^\flat) &= (\mu_z \circ T\mu_z \circ T^2 g) \circ Tf \\ G_T(g^\flat) \circ G_T(f^\flat) &= (\mu_z \circ Tg \circ \mu_y) \circ Tf, \end{aligned}$$

which does hold thanks to the naturality (and the associative law) of μ. □

Lemma 4. Provided with a monad $\langle T, \eta, \mu \rangle$ on X, $\langle F_T, G_T \rangle : X \to X_T$ defines the adjoints that induce the monad.

Proof. We see that the monad unit η defines the adjoints unit if we observe that η is a natural transformation $I_X \dot{\to} G_T F_T$ that gives an universal arrow $x \to Tx$ on $(x \downarrow G_T)$ for each $x \in X$. Precisely for each $y \in X$ and $f : x \to G_T y = Ty$, we have the following commutative diagram:

$$\begin{array}{ccc} & & Ty \\ & \overset{f}{\nearrow} & \uparrow {\scriptstyle \exists ! G_T f^\flat = \mu_y \circ Tf} \\ x & \xrightarrow[\eta_x]{} & Tx. \end{array}$$

Because $(\eta_x)^\flat$ is the composite identity of X_T and $(\mu_y \circ Tf \circ \eta_x)^\flat = f^\flat \circ (\eta_x)^\flat$, this indeed commutes using the hom-set correspondence $\hom_X(x, Ty) \cong \hom_{X_T}(x_T, y_T)$. The naturality holds automatically and hence $\langle F_T, G_T \rangle$ is an adjoints.

The counit $\epsilon_T : F_T G_T \dot{\to} I_{X_T}$ of the adjoints admits the component $(Tx)_T \to x_T$ for each $x_T \in X_T$ that corresponds to $Tx \to Tx$ in X. We see that $(\mathrm{id}_{Tx})^\flat$ is the universal arrow on $(F_T \downarrow x_T)$ when we examine the universality as we did for η.

The multiplication of the (adjoints-)induced monad is given by $\mu_T = G_T \epsilon_T F_T$. We see $(\mu_T)_x : T^2 x \to Tx$ gives the components of μ_T for each $x \in X$ by definition of F_T and G_T.

According to the definition of horizontal composition, namely:

$$X \overset{F_T}{\underset{F_T}{\rightrightarrows}} X_T \overset{F_T G_T}{\underset{I_{X_T}}{\rightrightarrows}} X_T \overset{G_T}{\underset{G_T}{\rightrightarrows}} X,$$

we also see that μ_T suffices the associative law and right (left) unit law; therefore μ_T coincides with μ. □

Proposition 1. Provided with a monad $\langle T, \eta, \mu \rangle$ on X together with an arbitrary adjoints $\langle F, G \rangle : X \to A$ (over X), we have the unique *Kleisli* comparison functor $L : X_T \to A$ whose "image" LX_T is a full subcategory of A. The objects of LX_T are comprised of Fx for all $x \in X$.

Proof. We denote by $\langle F, G, \eta, \epsilon \rangle : X \to A$ an adjoint and by $T = \langle GF, \eta, G\epsilon F \rangle$ its defining monad on X. The *Kleisli* comparison functor is characterized by the properties $GL = G_T$ and $LF_T = F$ or equivalently by the following commutative diagram (where double arrows are meant to chase respectively):

$$\begin{array}{ccc} A & \xleftarrow{L} & X_T \\ {\scriptstyle F}\uparrow\downarrow{\scriptstyle G} & & {\scriptstyle F_T}\uparrow\downarrow{\scriptstyle G_T} \\ X & = & X \end{array}$$

Chasing from the bottom right corner, the object function of L must be $Lx_T = Fx$.

Chasing from the top right corner, a possible candidate for the morphism function is given by $L(x_T \xrightarrow{f^b} y_T) = (\epsilon F)_y \circ Ff$ since we have $GLf^b = G_T f^b = \mu_y \circ Tf = (G\epsilon F)_y \circ GFf = G((\epsilon F)_y \circ Ff)$.

To confirm this is a functor, we need to show that L respect a composition, namely

$$L(g^b \circ f^b) = L(g^b) \circ L(f^b), \quad \forall (x_T \xrightarrow{f^b} y_T \xrightarrow{g^b} z_T) \in \mathrm{Mor}(X_T). \tag{1}$$

By definition of L and the *Kleisli* composition, this is done by showing the following diagram commute:

$$\begin{array}{ccccc}
Fx & \xrightarrow{Ff} & FTy & \xrightarrow{FTg} & FT^2z & & \\
 & & \downarrow{\scriptstyle (\epsilon F)_y} & & \downarrow{\scriptstyle F\mu_z} & & \\
 & & Fy & \xrightarrow[Fg]{} & FTz & \xrightarrow[(\epsilon F)_z]{} & Fz.
\end{array}$$

By seeing $F\mu$ as a horizontal composition of the natural transformations, we have:

$$F\mu_z = (F\mu)_z = (FG\epsilon F)_z = FG(\epsilon F)_z$$

Furthermore, we want to exploit the naturality of ϵF to admit the commutativity where the square in the above diagram is seen to be a 2-morphism from the component $(\epsilon F)_y$ over y to $(\epsilon F)_{Tz}$ over Tz.

Hence we are done if we show that the following extended diagram commutes:

$$\begin{array}{ccc}
FT^2z & \xrightarrow{(\epsilon FGF)_z} & FTz \\
{\scriptstyle F\mu_z = (FG\epsilon F)_z}\downarrow & & \downarrow{\scriptstyle (\epsilon F)_z} \\
FTz & \xrightarrow[(\epsilon F)_z]{} & Fz
\end{array}$$

This indeed commutes by definition of $\epsilon\epsilon F$.

Finally we'll see the uniqueness of L from the diagram below:

$$\begin{array}{ccccc}
A & \xrightarrow{G} & X & \xrightarrow{F} & A \\
{\scriptstyle L}\uparrow & & \| & & {\scriptstyle L}\uparrow \\
X_T & \xrightarrow{G_T} & X & \xrightarrow{F_T} & X_T
\end{array}
\qquad
\begin{array}{ccccc}
A(Fx, Fy) & = & A(Fx, Ly_T) & \xrightarrow{\phi} & X(x, GLy_T) \\
 & & {\scriptstyle L}\uparrow & & \| \\
X_T(x_T, y_T) & = & X_T(F_Tx, y_T) & \xrightarrow{\phi_T} & X(x, G_Ty_T)
\end{array}$$

The left diagram commutes by the properties defining L. On the right diagram, horizontal arrows are bijections by given adjoints and hence L is unique up to isomorphism.

By construction, the image LX_T is a full subcategory $F_X \subset A$ whose objects are comprised of Fx for each $x \in X$.

□

References

[1] Saunders MacLane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262.

[2] Eugenio Moggi. "Notions of computation and monads". In: *Information and Computation* 93.1 (1991). Selections from 1989 IEEE Symposium on Logic in Computer Science, pp. 55–92. ISSN: 0890-5401. DOI: https://doi.org/10.1016/0890-5401(91)90052-4. URL: https://www.sciencedirect.com/science/article/pii/0890540191900524.