# When a function on a cartesian product induces a "recovery" of the value that respects the per-component contributions?

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On a context of data modeling, we sometimes run into a problem of (vertical) schema decomposition where the future modification is expected either not to affect or at least respect for the existent data.

In a classic form, if the "contribution" of the schema modification is algebraically calculable with respect to the pre-existent schema; in other words, if there exists a surjective homomorphism of $k$-algebra $\nu : k[t_1, \ldots, t_m] \to k[t'_1, \ldots, t'_n]$ over a field $k$, the problem is canonically solved by giving the transformation $t \mapsto E_t(\nu)$, where we denote by $t$ and $t'$ the variables to be assigned the value of corresponding schema attributes and $E$ evaluates the data value $\langle t, \nu \rangle \to \nu_1(t), \ldots, \nu_n(t)$ at $t = (t_1, \ldots, t_m)$. Without loss of generality, some minor modification can be applied; for instance the division by a certain attribute is realized by the localization, i.e. an addition of fractional generator to $t$.

Within a real business system, however, we rarely find such rigid algebraic relation among the data attributes and hence the previous argument is barely applicable.

To think of a way of broadening the scope of domain (to unstructured Set), it may sound convincing to study the compatible conditions of evaluation map on each attribute. Simply put, we consider the following situation.

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**Proposition 1.** For finite sets $X_1, X_2$, we are given a function $h : X_1 \times X_2 \to V$ that takes value on $V$ (a discrete subset of $\mathbb{R}$). When and under what conditions there exist the unique pair of maps $h_i : X_i \to V_i$ on each factor that recover the value of $h$, namely;

$$\begin{array}{ccccc} V_1 & \overset{}{\longleftarrow\cdots} & V & \cdots\longrightarrow & V_2 \\ {\scriptstyle \exists! h_1}\uparrow & & {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \exists! h_2} \\ X_1 & \xleftarrow[p_1]{} & X_1 \times X_2 & \xrightarrow{p_2} & X_2. \end{array}$$

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This is precisely the pushout of $X_i \xleftarrow{p_i} X_1 \times X_2 \xrightarrow{h} V$ that gives the universal way of factoring out the contribution of $h_i$, which is not really practical in terms of "recovering" the value of $h$.

To make it somewhat practical, an obvious assumption would be to assign fixed points for each $X_i$; precisely, when $X_i$ are pointed (in a category endowed with the terminal object $*$), some chosen fixed (or default) values $* \to X_i \to V$ induce the desired pair of maps that commutes the following diagram:

$$\begin{array}{ccccc} V & = & V & = & V \\ {\scriptstyle h\circ i_1}\uparrow & & {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h\circ i_2} \\ X_1 & \overset{i_1}{\rightleftarrows} & X_1 \times X_2 & \overset{i_2}{\leftrightarrows} & X_2. \end{array}$$

Nevertheless, it is often improper or even impossible for some business requirements to give per-attribute (default) evaluation. This arises for instance when the value is determined by a tuple of attributes, not by single.

This argument is similar when we require $X_i$ belong to abelian category where a product is isomorphic to its coproduct [1].

Then, what would be a reasonable definition of "the recovery of the value of $h$"?

It may be happy to hear that there is a map $V_1 \times V_2 \to V$ that is compatible with the diagram in the above proposition, in which we immediately see that $V$ is isomorphic to $V_1 \times V_2$ and hence $h = h_1 \times h_2$; whereas it may imply some emergence of non-trivial classification (of the recovery) when we see the existence of the compatible map from direct product as "the most trivial".

In this article, we save for later the further detail of what would be a nice "fibration" $E \to B$ along with a map $V \to B$ to induce a "recovery" $E^*V \to V$ of $h : X_1 \times X_2 \to V$, which may give an idea of studying the proposition.

## References

[1] Miodrag Cristian Iovanov. *When is the Product isomorphic to the Coproduct*. 2006. arXiv: `math/0605112 [math.CT]`.