Nerve and realization - basic examples and adjointness

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Introduction

In what follows, we restrict the arguments to small (i.e. internal to **Set**) categories.

Definition. The **simplex category** Δ is a category whose objects are all finite ordinal numbers denoted by $[n]$ and the morphisms are all the weakly monotone functions between the ordinals.

Definition. A **simplicial object (in C)** is a functor $\Delta^{op} \to C$ for a category C. Dually, a **cosimplicial object (in C)** is a functor $\Delta \to C$.

Given a cosimplicial object Δ_C in a (cocomplete) category C, we have a pair of functors:

$$\mathbf{Set}^{\Delta^{op}} \underset{\mathcal{N}}{\overset{|\text{-}|}{\rightleftarrows}} C,$$

where we call |-| the *realization* and $\mathcal{N}$ the *nerve*.

The *realization* is an operation to build up a certain object along with the given building block according to a combinatorial design. This came originally of a geometric construction where the building block is standard (affine) simplex and the design is abstract simplicial complex. Conversely — on some account, the *nerve* operation introduces the given object combinatorics and dimensionality that are responsible for classifying the object.

Throughout the text, we focus on introducing the basics such as definitions and canonical examples while a circuitousness is carefully avoided. At the end, we'll show that the *realization* is a left adjoint to the *nerve* operation, owning to the fact that a presheaf on a small category is expressed as the colimit of representable presheaves.

Realization

Definition. Let C be a cocomplete category. For each simplicial set $S : \Delta^{op} \to \mathbf{Set}$ and cosimplicial object Δ_C in C, there is a functorial operation called the **tensor product (of functors)**:

$$\otimes : \mathbf{Set}^{\Delta^{op}} \times C^{\Delta} \to C,$$

defined by $S \otimes \Delta_C = \int^{n\in\Delta} S_n \cdot \Delta_{Cn}$, where $\int^{n\in\Delta}$ is the coend and $S_n \cdot \Delta_{Cn}$ is the copower $\sqcup_{\sigma\in S_n} \Delta_{Cn}$ in C.

Lemma 1. The tensor product $\langle S, \Delta_C\rangle \mapsto S \otimes \Delta_C$ of functors is functorial.

Proof. We are enough to show that for each natural transformations $\tau : S \dot{\to} S'$ and $\nu : \Delta_C \dot{\to} \Delta'_C$, the following canonical diagram commutes:

$$\begin{array}{ccc} \int^{n\in\Delta}(S_n)\cdot(\Delta_{Cn}) & \xrightarrow{\tau\otimes 1} & \int^{n\in\Delta}(S'_n)\cdot(\Delta_{Cn}) \\ \downarrow{\scriptstyle 1\otimes\nu} & & \downarrow{\scriptstyle 1\otimes\nu} \\ \int^{n\in\Delta}(S_n)\cdot(\Delta_{C}{}'_n) & \xrightarrow{\tau\otimes 1} & \int^{n\in\Delta}(S'_n)\cdot(\Delta_{C}{}'_n), \end{array}$$

where $\tau \otimes 1$ and $1 \otimes \nu$ are morphisms in C induced by the corresponding universal cowedges. To write these down, we see that for each (weakly) monotone function $[n] \to [m]$, there are universal cowedges:

$$\begin{array}{ccc} (S_n)\cdot(\Delta_{Cn}) & \xrightarrow{\tau_n\cdot\Delta_{Cn}} & (S'_n)\cdot(\Delta_{Cn}) \\ \downarrow & & \downarrow \\ \int^{n\in\Delta}(S_n)\cdot(\Delta_{Cn}) & \overset{\exists!\tau\otimes 1}{\dashrightarrow} & \int^{n\in\Delta}(S'_n)\cdot(\Delta_{Cn}) \\ \uparrow & & \uparrow \\ (S_m)\cdot(\Delta_{Cm}) & \xrightarrow{\tau_m\cdot\Delta_{Cm}} & (S'_m)\cdot(\Delta_{Cm}) \end{array}
\qquad
\begin{array}{ccc} (S'_n)\cdot(\Delta_{Cn}) & \xrightarrow{S'_n\cdot\nu_n} & (S'_n)\cdot(\Delta_{C'n}) \\ \downarrow & & \downarrow \\ \int^{n\in\Delta}(S'_n)\cdot(\Delta_{Cn}) & \overset{\exists!1\otimes\nu}{\dashrightarrow} & \int^{n\in\Delta}(S'_n)\cdot(\Delta_{C'n}) \\ \uparrow & & \uparrow \\ (S'_m)\cdot(\Delta_{Cm}) & \xrightarrow{S'_m\cdot\nu_m} & (S'_m)\cdot(\Delta_{C'm}). \end{array}$$

Since the copower $\cdot : \mathbf{Set}\times C \to C$ induced for each $n\in\Delta$ is functorial, the universality shows that the first diagram commutes. □

Definition. Given a cosimplicial object Δ_C in a cocomplete category C, a functor |-| defined by

$$|\text{-}| = (\text{-})\otimes\Delta_C : \mathbf{Set}^{\Delta^{op}} \to C$$

is called the **realization** (with respect to Δ_C).

Note. There is another (but equivalent) interpretation of realization as the left *Kan* extension $\mathrm{Lan}_{Y^\Delta}\Delta_C$ of Δ_C along the Yoneda embedding $Y^\Delta : \Delta \to \mathbf{Set}^{\Delta^{op}}$.

Example. (free R-module) For each $X\in\mathbf{Set}$, the assignment $[n]\mapsto X^n$ of the direct (i.e. Cartesian) product defines a simplicial set $X^{(-)} : \Delta^{op}\to\mathbf{Set}$ equipped with the obvious face maps and degenerate maps such as $\partial_1\langle x_0,x_1\rangle = x_0$ and $s_1\langle x_0,x_1\rangle = \langle x_0,x_1,x_1\rangle$, respectively. Similarly for each ring R, the assignment $[n]\mapsto R^{\oplus n}$ of the free R-module gives the covariant functor $R_{(-)} : \Delta\to\mathbf{Mod}_R$, where the coface maps and codegenerate maps look like $\delta_1\langle a_0,a_1\rangle = \langle a_0,0,a_1\rangle$ and $\sigma_1\langle a_0,a_1,a_2\rangle = \langle a_0,a_1+a_2\rangle$, respectively.

We claim that there is an isomorphism $\iota : \displaystyle\int^n X^n\cdot R_n \to \bigoplus_{x\in X} Rx$ of R-module from the realization $|X^{(-)}|$ with respect to $R_{(-)}$ to the free R-module generated by X.

To see this, note that the R-module homomorphisms $\lambda_n : X^n\cdot R_n \to \bigoplus_{x\in X} Rx$ that sends $\langle a_0,\dots,a_{n-1}\rangle_{\mathbf{x}}$ to $\sum_{i<n} a_i x_i$ for each $n\in\Delta$ and $\mathbf{x} = \langle x_0,\dots,x_{n-1}\rangle\in X^n$ is (a component of) cowedge, which fits into the diagram:

$$\begin{array}{ccccc} X^n\cdot R_m & \xrightarrow{(X^f\cdot R_m)^*} & X^m\cdot R_m & & \\ \downarrow{\scriptstyle (X^n\cdot R_f)_*} & & \downarrow & \searrow{\scriptstyle \lambda_m} & \\ X^n\cdot R_n & \longrightarrow & \int^n X^n\cdot R_n & \overset{\iota}{\dashrightarrow} & \bigoplus_{x\in X} Rx, \\ & & {\scriptstyle \lambda_n :\ X^n\cdot R_n \to \bigoplus_{x\in X} Rx} & & \end{array}$$

where $f : [m]\to[n]$ is an arbitrary morphism of Δ.

λ being a cowedge encodes an expression of each element of $\bigoplus_{x\in X} Rx$ in terms of the linear combination with the coefficients in R, namely $\forall v\in\bigoplus_{x\in X} Rx, \exists a_i\in R, x_i\in X$ s.t. $v = \sum_i a_i x_i$. This can be seen explicitly by chasing the cowedge for some structure maps. For example, let $f : [2]\to[4]$ be a coface defined by $[0,1]\mapsto[0,\hat{1},2,\hat{3}]$, assigning 0 to 0 and 1 to 2 (i.e. m=2, n=4). An element $\langle a_0,a_1\rangle_{[x_0\dots x_3]}\in X^4\cdot R_2$ is passed along the diagram in the following way:

$$\begin{array}{ccc} \langle a_0,a_1\rangle_{[x_0\dots x_3]} & \longmapsto & \langle a_0,a_1\rangle_{[x_0 x_2]} \\ \downarrow & & \downarrow \\ \langle a_0,0,a_1,0\rangle_{[x_0\dots x_3]} & \longmapsto & a_0x_0 + a_1x_2. \end{array}$$

For the case of another structure map is similar. When $f : [4]\to[2]$ is a codegenerate map $[0,1,2,3]\mapsto[0,1]$ that assigns 0,1 to 0 and 2,3 to 1 (i.e. m=4, n=2), the corresponding element chasing is given by:

$$\begin{array}{ccc} \langle a_0,\dots,a_3\rangle_{[x_0x_1]} & \longmapsto & \langle a_0,\dots,a_3\rangle_{[x_0x_0x_1x_1]} \\ \downarrow & & \downarrow \\ \langle a_0+a_1, a_2+a_3\rangle_{[x_0x_1]} & \longmapsto & (a_0+a_1)x_0 + (a_2+a_3)x_1. \end{array}$$

These observations show that the number of generating element is reduced at most the number of coefficients in order to represent an element in the free module, and on the other way around; namely the number of coefficients is reduced to at most the number of generating element in order to represent the element.

ι is shown to be an isomorphism by induction of cardinal of X.

Example. (geometric realization) Let $\Delta_{\mathbf{Top}} : \Delta \to \mathbf{Top}$ be a cosimplicial object in **Top** that assigns the standard (affine) n-simplex for each $[n] \in \Delta$. The coface and codegenerate maps are obvious one.

The realization $|\text{-}| : \mathbf{Set}^{\Delta^{op}} \to \mathbf{Top}$ with respect to $\Delta_{\mathbf{Top}}$ is called **geometric realization**.

Note. Consider a simplicial set S that is realized to singleton. When $n > m$, each codegenerate map $[n+1] \to [m+1]$ induces degenerate map $S_m \to S_n$ as a distinct morphism in **Set** if $S_m \neq \varnothing$. By assumption, we must have $S_0 \cdot \Delta_0 = \{e_0\} \neq \varnothing$ hence we have non-empty sets of higher simplices. The resulting simplicial set is given as following (e_i^d is a degenerate simplex that is to be collapse to lower dimensional simplex): Despite that the realization turns out to be one point space, the identification

Table 1: sets of k-simplices for the simplicial set structure on the singleton space

S_0	S_1	S_2	S_3	$\dots$
e_0	e_1^d	e_2^d	e_3^d	$\dots$

procedures should be taken carefully. If we are not taking degenerate maps into account, we have an infinite sequence of dunce-hat-like spaces glued all together that is grown in complexity as n increases.

The minimal simplicial set structure on S^1 is in the similar manner except that non-degenerate operation glues the endpoints of 1-simplex to the only 0-simplex. As a delta set, the sets of simplices are comprised of one 0-simplex and one 1-simplex, whereas a simplicial set is given in the following (e_i^d indicates again a degenerate simplex): Only difference from the singleton is that e_1 is added to X_1 and

Table 2: sets of k-simplices for the simplicial set structure on S^1

X_0	X_1	X_2	X_3	$\dots$
e_0	e_1, e_1^d	e_2^d	e_3^d	$\dots$

this is necessary for a distinct non-collapsing 1-simplex to build up the interior of a circle. We can see the degenerate operation collapsing e_1^d only as prescribed way in dimension one:

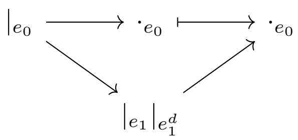
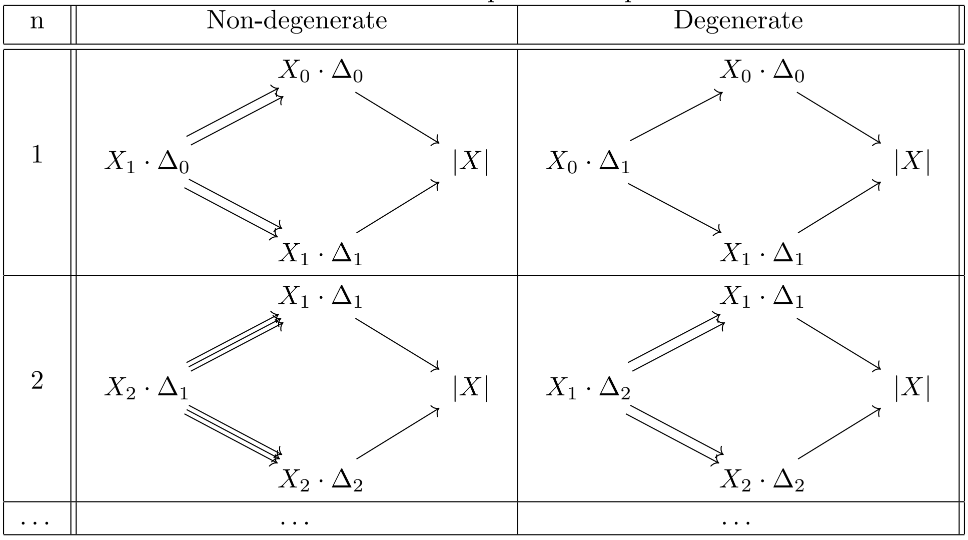


Table 3: structure maps for a simplicial set



Nerve

Definition. Given a cosimplicial object Δ_C in a category C, a functor defined by the composite:

$$\mathcal{N} = C(\Delta_C, -) : C \to \mathbf{Set}^{\Delta^{op}}$$

of Δ_C followed by the (contravariant) Yoneda embedding is called **nerve** (with respect to Δ_C).

Example. (**Cat** nerve) Let **Cat** be a category of small categories together with the morphisms of all the functors.

Because a finite totally ordered set can be interpreted as a (skeletal thin) category (i.e. as a full sub-category $\Delta \ni [n] \mapsto \mathbf{n} \in \mathbf{Cat}$), we can take a cosimplicial object $\Delta_{\mathbf{Cat}}$ that assigns a finite totally ordered set with $(n+1)$-elements $\mathbf{n} \in \mathbf{Cat}$ to each ordinal $[n] \in \Delta$. For an arbitrary category D, each element of $\mathbf{Cat}(\Delta_{\mathbf{Cat}n}, D)$ specifies a sequence of n-composable morphisms

$$*_0 \xrightarrow{f_1} *_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} *_{n-1} \xrightarrow{f_n} *_n$$

in D.

The structure maps are given as follows:

$$\begin{aligned}
&d_0\langle f_1, \dots, f_n\rangle = \langle f_2, \dots, f_n\rangle,\ d_n\langle f_1, \dots, f_n\rangle = \langle f_1, \dots, f_{n-1}\rangle \\
&d_i\langle f_1, \dots, f_n\rangle = \langle \dots, f_{i+1} \circ f_i, \dots\rangle \quad (0 < i < n), \\
&\\
&s_0\langle f_1, \dots, f_n\rangle = \langle id_{*_0}, f_1, \dots, f_n\rangle,\ s_n\langle f_1, \dots, f_n\rangle = \langle f_1, \dots, f_n, id_{*_n}\rangle \\
&s_i\langle f_1, \dots, f_n\rangle = \langle f_1, \dots, f_i, id_{*_i}, f_{i+1}, \dots, f_n\rangle \quad (0 < i < n).
\end{aligned}$$

Example. (group nerve) Since a group is interpreted as a category of single object equipped with all the end functions having inverses as the morphisms, we can think of group nerve in terms of **Cat** nerve. The properties of nerve are often well understood by the composition of realization. In fact, $|\mathcal{N}G|$ is known as a $K(G, 1)$ space for arbitrary group G, i.e. the class of spaces whose fundamental group is isomorphic to G.

Let G be an infinite cyclic group (hence we write additively by virtue of $G \cong \mathbb{Z}$). The zero skeleton $|\mathcal{N}G_0|$ is a point because the group has single object $*$ in **Cat**. The one skeleton $|\mathcal{N}G_1|$ is a bouquet of infinite circles where each circle corresponds to an integer while 0 is in the image of $s_0\langle *\rangle = \langle id_*\rangle = \langle +0\rangle$ hence degenerate. The two skeleton $|\mathcal{N}G_2|$ is realized by tarping all the triples of circles labelled n, m and $n+m$ (duplicate integers allowed) with a standard 2-simplex according to the labels. For example a two simplex represented by $\langle 1, 1\rangle$ is realized by glueing 2 edges to the circle labelled 1 and the 1 edge to the circle labelled 2, and so on. The higher skeletons are similar but the complexity as a space grows rapidly.

When $G = \mathbb{Z}_2$, $|\mathcal{N}G_0| = *$ and $|\mathcal{N}G_1| = S^1$. This is because every circles labelled with even numbers degenerate to the only vertex and those with odd numbers are identified with single circle. The only non-degenerate simplex in $\mathcal{N}G_2$ is of the form $\langle 1, 1\rangle$ that is realized by collapsing an edge of standard 2-simplex to the point and the rest of two edges are glued to the only circle with the deduced direction, in opposite. We conclude that $|\mathcal{N}G_2| = \mathbb{R}P^2$. Similar arguments show that $|\mathcal{N}G_n| = \mathbb{R}P^n$ and $|\mathcal{N}G| = \mathbb{R}P^\infty$.

Lemma 2. Let J be a small category. Any presheaf on J is canonically the colimit of representable presheaves.

Proof. Let $F : J^{op} \to \mathbf{Set}$ be an arbitrary presheaf on J. For each $j \in J$, we denote by $h^j = \hom(-, j) \in \mathbf{Set}^{J^{op}}$ the (representable) contravariant hom-functor. Let $\mathbf{Rep}_F$ be a full subcategory of $(\mathbf{Set}^{J^{op}} \downarrow F)$ where the objects are all the natural transformations (from the representable presheaves) $h^j \dot{\to} F$ and the morphisms are the induced natural transformations $h^\gamma_* : h^j \dot{\to} h^k$ for each $(\gamma : j \to k) \in J$ such that the following diagram commutes:

$$\begin{array}{ccc} h^j & \xrightarrow{h^\gamma_*} & h^k \\ & \searrow \quad \swarrow & \\ & F & \end{array}.$$

We denote the restricted projection of comma category by $\phi : \mathbf{Rep}_F \to \mathbf{Set}^{J^{op}}$, namely $\phi(h^j \to F) = h^j$. By definition, each object $c \in \mathbf{Rep}_F$ corresponds (not necessarily uniquely) to a (component of) cocone $\phi(c) \dot{\to} F$ to F which yields the unique morphism of presheaf $\kappa : \varinjlim_{h^j \to F} h^j \dot{\to} F$ since the category of presheaves is cocomplete.

We claim that κ is a natural isomorphism.

First of all, κ is natural in the sense that each $(\gamma : k \to j) \in J$ yields the commutative square of universal cocones:

$$\begin{array}{ccc} Fj & \xrightarrow{(F\gamma)^*} & Fk \\ \uparrow{\scriptstyle \kappa_j} & & \uparrow{\scriptstyle \kappa_k} \\ \varinjlim_{h^j \to F} h^j j & \dashrightarrow & \varinjlim_{h^j \to F} h^j k \\ \uparrow & & \uparrow \\ \phi(c)j & \xrightarrow{h^*_\gamma} & \phi(c)k. \end{array} \qquad (c_j : \phi(c)j \to Fj,\ c_k : \phi(c)k \to Fk)$$

For each $j \in J$ and $x \in Fj$, we can find some $(c : h^j \dot{\to} F) \in \mathbf{Rep}_F$ such that $c_j : \phi(c)j \to Fj$ sends 1_j to x hence κ is surjective on each component.

On the other hands, assuming $c_j(\lambda) = c'_j(\lambda')$ for some $\lambda \in \phi(c)j$ and $\lambda' \in \phi(c')j$, then it is immediate to see that $\lambda = \lambda'$ in $\varinjlim_{h^j \to F} h^j j$.

$$\begin{array}{ccc} \varinjlim_{h^j \to F} h^j j & \xrightarrow{\kappa_j} & Fj \\ \uparrow & \nearrow \nwarrow{\scriptstyle c_j} & \uparrow{\scriptstyle c'_j} \\ \phi(c)j & \xrightarrow{h^\gamma_*} & \phi(c')j. \end{array}$$

∎

Note. We denoted by $F \simeq \varinjlim_{h^j \to F} h^j$ a presheaf F expressed as the colimit of representable presheaves on $\mathbf{Rep}_F$ despite that $F \simeq \varinjlim_{c \in \mathbf{Rep}_F} \phi(c)$ may be rigorously expected notation.

Lemma 3. Let J be a small category. Given a copresheaf $J_C : J \to C$ that takes value in cocomplete category C, a representable presheaf $h^j \in \mathbf{Set}^{J^{op}}$ is canonically realized to $J_C(j)$ in a sense that there is an isomorphism in C:

$$|h^j| = \int^{k \in J} h^j k \cdot J_C(k) \to J_C(j).$$

This can be thought of as a variation of so called *co-Yoneda lemma.*

Proof. For any $Y \in C$ and $j \in J$, the following equation holds:

$$
\begin{array}{rcll}
C(|h^j|, Y) & = & C(\int^{k \in J} h^j k \cdot J_C(k), Y) & \\
& \simeq & \int_{k \in J} C(h^j k \cdot J_C(k), Y) & \because \text{a coend property as a colimit} \\
& \simeq & \int_{k \in J} \mathbf{Set}(h^j k, C(J_C(k), Y)) & \because \text{the adjoint of copower} \\
& \simeq & \mathbf{Set}(h^j, C(J_C, Y)) & \because \text{by definition of end} \\
& \simeq & C(J_C(j), Y). & \because \text{Yoneda lemma}
\end{array}
$$

Then $|h^j| \cong J_C(j)$ follows from argument involved with Yoneda embedding — $C(|h^j|, \text{-})$ and $C(J_C(j), \text{-})$ are isomorphic as representable copresheaves on C in the sense that there is a natural isomorphism τ between them to which the unique isomorphism corresponds.

$$
\begin{array}{ccccc}
C & & |h^j| & \underset{\exists !}{\overset{\exists !}{\rightleftarrows}} & J_C(j) \\
\downarrow \mathbf{y} & & \downarrow & & \downarrow \\
\mathbf{Set}^C & & C(|h^j|, \text{-}) & \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} & C(J_C(j), \text{-})
\end{array}
$$

□

Proposition 1. Let C be a cocomplete category. When a cosimplicial object $\Delta_C : \Delta \to C$ is given, $\langle |\text{-}|, \mathcal{N} \rangle : \mathbf{Set}^{\Delta^{op}} \to C$ is an adjoint pair.

Proof.

$$
\begin{array}{rcll}
C(|S|, Y) & \simeq & C(|\varinjlim_{h^n \to S} h^n|, Y) & \because \textbf{Lemma 2} \\
& \simeq & C(\varinjlim_{h^n \to S} |h^n|, Y) & \because \text{a property of coend as colimit} \\
& \simeq & \varprojlim_{h^n \to S} C(|h^n|, Y) & \because \text{hom-colimit exchange} \\
& \simeq & \varprojlim_{h^n \to S} C(\Delta_C n, Y) & \because \textbf{Lemma 3} \\
& = & \varprojlim_{h^n \to S} \mathcal{N}(Y) n & \\
& \simeq & \varprojlim_{h^n \to S} \mathbf{Set}^{\Delta^{op}}(h^n, \mathcal{N}(Y)) & \because \text{Yoneda lemma} \\
& \simeq & \mathbf{Set}^{\Delta^{op}}(S, \mathcal{N}(Y)) & \because \textbf{Lemma 2}
\end{array}
$$

□

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