Vector Bundles 2

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2 Vector Bundle

2.2 Sections

Proposition 1. an n-dimensional bundle $p : E \to B$ is isomorphic to the trivial bundle iff it has n sections $s_1, \dots, s_n$ such that the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$.

Proof. When $f : B \times \mathbb{R}^n \to E$ is an isomorphism, we can define such sections by $s_i(b) = f(b, v_i)$ for each i, where $v_i \in \mathbb{R}^n$ is a linearly independent i-th n-vector. The bundle isomorphism takes linearly independent sections to linearly independent sections.

On the other hand, when $\{s_i\}_i$ is such a series of sections, a bundle isomorphism can be defined by:

$$f : B \times \mathbb{R}^n \to E; \quad (b, t_1, \dots, t_n) \mapsto \sum_i t_i s_i(b).$$

This is an isomorphism on each fiber $p^{-1}(b)$ and furthermore, a homeomorphism. This is because its composition with a trivialization $h^{-1} : p^{-1}(U) \to U \times \mathbb{R}^n$ is continuous by definition of f and the topology on E, for point $(b, t) \in U \times \mathbb{R}^n$, there exists a transition function $g : U \to GL_n(\mathbb{R})$ such that $h^{-1}(f(b, t)) = (b, g(b) \cdot t)$ where $g(b)$ continuously depends on $s_i(b)$, hence on b.

$$\begin{array}{ccc} B \times \mathbb{R}^n \supset U \times \mathbb{R}^n & \xrightarrow{f} & p^{-1}(U) \subset E \\ & \searrow & \cong \downarrow h^{-1} \\ & & U \times \mathbb{R}^n \end{array}$$

Its inverse $(b, s) \mapsto (b, g(b)^{-1} \cdot s)$ is given by inverted determinant of $g(b)$ times its adjugate, which is again continuous. □

Note. In a literature such as [1], the proof is broken into two parts; the later employs a lemma asserting that fiber-wise isomorphic continuous function is homeomorphism, hence bundle isomorphism. Our vector bundle explicitly includes *coordinate transformation* as its definition, resulting that we know the composite function $h^{-1} \circ f$ is given by a regular matrix that continuously depends on the first coordinate.

Example. The tangent bundle $TS^1 \to S^1$ is trivial since it admits non-vanishing global section

$$(x_1, x_2) \mapsto (-x_2, x_1).$$

Example. To see non-triviality of the tangent bundle over S^2, consult *Hairy Ball Theorem.*

2.3 Whitney Sum

Given two vector bundles $p_i : E_i \to B \quad (i = 1, 2)$ over the same base space, the *direct sum* (*Whitney sum*) of E_1 and E_2 is a space defined by:

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\},$$

or concisely the pullback of a diagram $E_1 \xrightarrow{p_1} B \xleftarrow{p_2} E_2$. This space indeed is a vector bundle with the fiber $p_1^{-1}(b) \oplus p_2^{-1}(b)$ over $b \in B$, which is linearly isomorphic to $\mathbb{R}^{n_1+n_2}$. The local trivialization is given by:

$$h_1 \oplus h_2 : U \times (\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}) \to p_1^{-1}(U) \oplus p_2^{-1}(U),$$

where $h_1 \oplus h_2$ is the induced map from a pullback diagram:

$$\begin{array}{ccccc} & & U & & \\ & \overset{p_1}{\nearrow} & & \overset{p_2}{\nwarrow} & \\ p_1^{-1}(U) & \longleftarrow & p_1^{-1}(U) \oplus p_2^{-1}(U) & \longrightarrow & p_2^{-1}(U) \\ {\scriptstyle h_1}\uparrow & & \vdots & & \uparrow{\scriptstyle h_2} \\ U \times \mathbb{R}^{n_1} & \longleftarrow & U \times (\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}) & \longrightarrow & U \times \mathbb{R}^{n_2}. \end{array}$$

Because the inverse of $h_1 \oplus h_2$ is analogously induced and all maps in the diagram are continuous, this is a trivialization over U.

Example. *trivial bundles* As already stated, at least implicitly, two trivial bundles sum up to a trivial bundle by the direct sum.

Example. *stably trivial bundle* A vector bundle that becomes trivial bundle after taking the direct sum with a trivial bundle is called *stably trivial.* The tangent bundle TS^n over n-sphere is such example, by taking direct sum with normal bundle NS^n, which is isomorphic to a trivial bundle $S^n \times \mathbb{R}$. The isomorphism is given by:

$$f : TS^n \oplus NS^n \to S^n \times \mathbb{R}^{n+1}; \quad (x, v, tx) \mapsto (x, v + tx) \quad (x \perp v \text{ and } t \in \mathbb{R}).$$

2.4 Inner Products

Definition. A topological space X is called *paracompact* if for any open cover $\mathcal{U} = \{U_\alpha\}$ of X, there exists a locally finite open refinement of $\mathcal{U}$.

Definition. For a topological space X and given an open cover $\mathcal{U} = \{U_\alpha\}$, *a (continuous) partition of unity subordinated to the cover* $\mathcal{U}$ is a collection $\{u_j\}_{j \in J}$ of (continuous) functions $u_j : X \to [0, 1]$ s.t.

1. $\mathrm{Supp}(u_j) := \overline{u_j^{-1}((0, 1])} \subset U_\alpha$ for some α,
2. $\forall x \in X, \quad u_j(x) \neq 0$ for only finitely many $j \in J$,
3. $\forall x \in X, \quad \sum_j u_j(x) = 1$.

Fact. Let X be a Hausdorff space. Then X is paracompact iff for any open cover, X admits a (continuous) partition of unity subordinated to the cover.

Note. By definition, it implies that $\{u_j^{-1}((0, 1])\}_{j \in J}$ is an open refinement of original open cover $\mathcal{U}$, hence again an open cover.

Definition. Let $p_i : E_i \to B_i \quad (i = 1, 2)$ be two distinct vector bundles. A pair of continuous maps $\langle \hat{f} : E_1 \to E_2, f : B_1 \to B_2 \rangle$ is called a *bundle map* (or *bundle homomorphism*) if it commutes the diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2, \end{array}$$

where f restricts to a linear map $f|_x : p_1^{-1}(x) \to p_2^{-1}(f(x))$ on each fiber.

Note. When $B_1 = B_2$, we omit a map of base spaces and as such the fiber preserving continuous map $\hat{f} : E_1 \to E_2$ is then called a *bundle map* over B_1.

Definition. Let $p : E \to B$ be a vector bundle over a topological field k. A bundle map

$$\langle , \rangle : E \oplus E \to B \times k$$

over B to the trivial line bundle is called an *inner product* of E if the morphism restricts to a positive definite symmetric bilinear form

$$\langle , \rangle_x : p^{-1}(x) \oplus p^{-1}(x) \to k$$

on each fiber.

Definition. *vector subbundle* Given a vector bundle $p : E \to B$, a subspace $E_0 \subset E$ intersecting each fiber of E in a vector subspace such that the restriction $p|E_0 : E_0 \to B$ is a vector bundle, is called *vector subbundle* of E.

Proposition 2. An inner product exists for a vector bundle $p : E \to B$ if B is paracompact Hausdorff [1, Proposition 1.2].

Proof. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover composed of coordinate charts accompanying with a collection of coordinate functions $\phi_j : U_j \times k^n \to p^{-1}(U_j)$.

An inner product is locally defined without paracompactness, explicitly:

$$\langle , \rangle_j : E|_{U_j} \oplus E|_{U_j} = p^{-1}(U_j) \oplus p^{-1}(U_j) \xrightarrow{\phi_j^{-1} \oplus \phi_j^{-1}} U_j \times k^{2n} \xrightarrow{1 \times \langle , \rangle_{\mathbb{R}^n}} U_j \times k.$$

To extend this map to $E \oplus E$, notice that for any point $x \in U_i \cap U_j \neq \varnothing$, an non-empty intersection of two coordinate neighbourhoods, the values of local inner products differ by corresponding glueing function, explicitly:

$$\langle v_x, w_x \rangle_i = \langle g_{ij}(x) \cdot v_x, g_{ij}(x) \cdot w_x \rangle_j,$$

for some $g_{ij}(x) \in \mathrm{GL}_n(k)$. To amend this, take a partition of unity $\{u_\lambda\}_{\lambda \in \Lambda}$ subordinated to $\mathcal{U}$ and a function $\alpha : \Lambda \to J$ of indices such that $\mathrm{Supp}(u_\lambda) \subset U_{\alpha(\lambda)}$. Then the inner product defined by:

$$\langle v_x, w_x \rangle = \sum_\lambda u_\lambda(x) \langle v_x, w_x \rangle_{\alpha(\lambda)}$$

is the desired map. □

2.5 Tensor Products

Definition. Define the tensor product of vector bundles $E_1 \xrightarrow{p_1} B \xleftarrow{p_2} E_2$ by:

$$E_1 \otimes E_2 = \bigcup_{x \in B} (p_1^{-1}(x) \otimes p_2^{-1}(x)) \cong \bigcup_{x \in B} (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}),$$

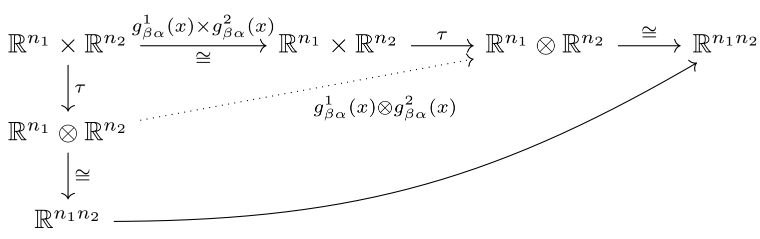
which as a set, is (disjoint) union of the tensor product of fibers. The topology is defined on each subset $p_1^{-1}(U) \otimes p_2^{-1}(U) \subset E_1 \otimes E_2$ for any coordinate chart $U \subset B$ on which E_i (i=1,2) are trivialized.

We can set a topology on the subset by letting the fiberwise isomorphic bijection

$$p_1^{-1}(U) \otimes p_2^{-1}(U) \ni v_x \otimes w_x \mapsto \langle x, v \otimes w \rangle \in U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$$

a homeomorphism, considering $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ as the space $\mathbb{R}^{n_1 n_2}$.

A transition function is given by $(x, v \otimes w) \sim (x, g^1_{\beta\alpha}(x) \otimes g^2_{\beta\alpha}(x)(v \otimes w))$ for each $(x, v \otimes w) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_1 n_2}$, with $g^1_{\beta\alpha} \otimes g^2_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n_1 n_2, \mathbb{R})$ mapping to the Kronecker product of two matrices:



Note. Transition functions $g^1_{\beta\alpha} \otimes g^2_{\beta\alpha}$ and $g^1_{\gamma\beta} \otimes g^2_{\gamma\beta}$ of tensor product satisfy cocycle condition for each $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Proposition 3. The set of isomorphism classes of vector bundle over a base space B, $\bigsqcup_{n \geq 0} \mathrm{Vect}^n(B)$ is a *commutative monoid* (Abelian group not necessarily with the inverse for each element) with respect to the tensor product. Furthermore, the tensor product is distributive with respect to the direct sum.

It is enough to show the following properties.

1. *associativity* $(E_1 \otimes E_2) \otimes E_3 \approx E_1 \otimes (E_2 \otimes E_3)$,

2. *commutativity* $E_1 \otimes E_2 \approx E_2 \otimes E_1$,

3. *identity* $E \otimes (B \times \mathbb{R}) \approx E$,

4. *distributive with respect to direct sum* $E_1 \otimes (E_2 \oplus E_3) \approx (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$,

Proof. We only show the distributive property here and the other properties will be omitted for detail. Define a distribution map by:

$$f : E_1 \otimes (E_2 \oplus E_3) \to (E_1 \otimes E_2) \oplus (E_1 \otimes E_3); \quad f((x,\alpha), v_1 \otimes (v_2, v_3)) = ((x,\alpha), v_1 \otimes v_2, v_1 \otimes v_3),$$

which is fiberwise linear isomorphism and continuous over each coordinate chart, say:

$$\begin{array}{ccc} p_1^{-1}(U_\alpha) \otimes (p_2^{-1}(U_\alpha) \oplus p_3^{-1}(U_\alpha)) & \xrightarrow{f} & (p_1^{-1}(U_\alpha) \otimes p_2^{-1}(U_\alpha)) \oplus (p_1^{-1}(U_\alpha) \otimes p_3^{-1}(U_\alpha)) \\ \approx\uparrow & & \approx\uparrow \\ U_\alpha \times (\mathbb{R}^{n_1} \otimes (\mathbb{R}^{n_2} \oplus \mathbb{R}^{n_3})) & \longrightarrow & U_\alpha \times ((\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}) \oplus (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_3})), \end{array}$$

so that f is a bundle isomorphism as in the **Note** after the **Proposition 1**. □

Corollary 1. $\mathrm{Vect}^1(B)$ is an Abelian group with respect to the tensor product.

Proof. For a line bundle $E \in \mathrm{Vect}^1(B)$, the inverse E^{-1} is defined by replacing the transition functions of E with its inverses. Then an isomorphism to the trivial line bundle is given by:

$$f : E \otimes E^{-1} \to B \times \mathbb{R}; \quad f((x,\alpha), [v \otimes w]) = (x, vw),$$

on each coordinate chart U_α. The equivalence class is taken effect of transition functions whenever $x \in U_\alpha \cap U_\beta$, in a form:

$$v \otimes w \sim g_{\beta\alpha}(x)v \otimes g_{\beta\alpha}^{-1}(x)w = g_{\beta\alpha}(x)g_{\beta\alpha}^{-1}(x)(v \otimes w) = v \otimes w.$$

Hence the bundle has trivial structure group, which makes f continuous. The transition functions commute since they are one dimensional (non-zero) real numbers at each point. □

References

[1] A. Hatcher. *Vector Bundles and K-Theory.* http://www.math.cornell.edu/˜hatcher. 2003.