Vector Bundles 3

stma

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2 Vector Bundle

2.6 Associated Fiber Bundles

Because a vector bundle is required to have fiber isomorphic to a vector space, it is unlikely to be applied to vast instances that naturally arise in geometry. Some topological theory, such as *Leray-Hirsch theorem*, can be applied only to more general object called *fiber bundle*, so we will describe how to construct a fiber bundle from a vector bundle. The constructed fiber bundle is called *associated (fiber) bundle* (to the given vector bundle).

Definition. A map $p : E \to B$ is called a *fiber bundle* over B with fiber F if we replace the model fiber $\mathbb{R}^n$ with F and isomorphism with homeomorphism in the definition of vector bundle.

Example. *(unit) sphere bundle* For a vector bundle $p : E \to B$ with the (model) fiber $\mathbb{R}^n$ equipped with an inner product, the *associated (unit) sphere bundle* $S(E)$ is defined by:

$$S(E) = \{(x, v) \in E : |v| = 1\},$$

with fiber S^{n-1}. The local trivialization is constructed by first giving an isometric local trivialization, or equivalently choosing an inner product on E so that the following diagram commutes (this inner product is compatible with the standard one in $\mathbb{R}^n$):

$$\begin{array}{ccc} U \times \mathbb{R}^{2n} & \xrightarrow{\langle , \rangle} & U \times \mathbb{R} \\ \downarrow{\scriptstyle h_U \oplus h_U} & & \| \\ p^{-1}(U) \oplus p^{-1}(U) & \xrightarrow{\langle , \rangle_U} & U \times \mathbb{R}. \end{array}$$

Secondly, the local trivialization restrict to unit vectors on each $x \in U$ gives the local trivialization on $S(E)$.

Note. Without inner product, the sphere bundle of E can be defined by removing zero section and factoring out scalar multiplication of positive real numbers on each fibers, namely:

$$S(E) = (E \backslash 0)/\mathbb{R}^{+},$$

analogously, the *disc bundle* $D(E)$ of E whose fiber has the length equal or less than 1, can be defined by the mapping cylinder of the projection $S(E) \to B$, precisely:

$$D(E) = ((S(E) \times I) \sqcup B)/\big((v_b, 0) \sim b\big).$$

The local trivialization is induced by restricting the original $U \times \mathbb{R}^n \to p^{-1}(U)$.

Example. *projective bundle* The *projective bundle* $P(E)$ of E is defined by the quotient space of $S(E)$ factoring out scalar multiplication of non-zero real numbers:

$$P(E) = S(E)/\mathbb{R}^{*}.$$

The local trivialization is induced by the quotient, homeomorphic to $U \times \mathbb{R}P^{n-1}$ over U.

Example. *flag bundle* Let $E \to B$ a vector bundle with fiber isomorphic to $\mathbb{R}^n$. As a generalization of projective bundle, for chosen $k \leq n$, the (k-folded) *flag bundle* is a subspace $F_k(E)$ of $\prod^k P(E)$ consisting of k-tuples of orthogonal lines in fibers of E (this product is taken as the pullback of $E_1 \to B \leftarrow E_2$ within fiber bundles, analogous to Whitney sum in vector bundles).

The local trivialization is homeomorphism of $F_k(U)$, the subspace of $U \times \prod^k \mathbb{R}P^{n-1}$ on U, composed of k-tuples of orthogonal lines on each fiber.

Example. *Stiefel bundle* The *Stiefel bundle* $V_k(E) \to B$, having k-tuples of orthogonal unit vectors in fibers of E as the point, is though of as a subspace $V_k(E) \subset \prod^k S(E)$. Stiefel bundle is similar object to flag bundle, yet it takes distinct form in that the linear subspace generated by orthonormal k-frames on each fiber is given an algebraic interpretation.

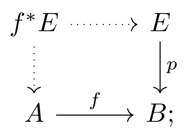
Example. *Grassmann bundle* The *Grassmann bundle* $G_k(E) \to B$ of k-dimensional linear subspaces of fibers of E, is the quotient space of $V_k(E)$ obtained by $GL_k(\mathbb{R})$ action on the fiber, identifying two k-frames in a fiber if they span the same subspace of the fiber. The fiber of $G_k(E)$ is the Grassmann manifold $G_k(\mathbb{R}^n)$ of k-planes through the origin in $\mathbb{R}^n$.

2.7 Pullback Bundles and its classifying property

Definition. (pullback bundle) Given a vector bundle $p : E \to B$ and a map $f : A \to B$, *the pullback bundle* f^*E (of E along f) is a space defined by

$$f^*E = \{(a, v_b) \in A \times E | f(a) = p(v_b) = b\}.$$

Fact. The pullback bundle f^*E of $A \xrightarrow{f} B \xleftarrow{p} E$ is a vector bundle that commutes the diagram



furthermore, the pullback bundle is unique (against f and p) up to isomorphism.

Proposition 1. For given maps between (base) spaces, the induced bundle maps satisfy the following properties:

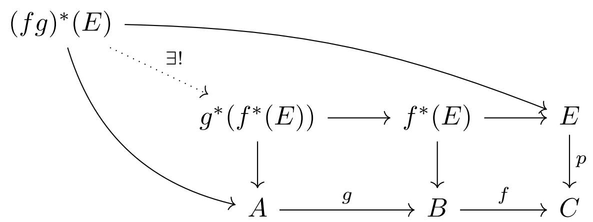
(1) $(fg)^*(E) \approx g^*(f^*(E))$;

(2) $\imath^*(E) \approx E$;

(3) $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$;

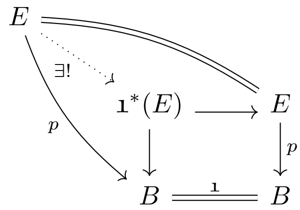
(4) $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$.

Proof. (1) By the universal property of pullbacks, the following diagram shows that the fiber-wise isomorphic bundle map over A (i.e. dotted arrow) is uniquely induced up to isomorphism, which turns out to be a bundle isomorphism.



□

Proof. (2) The pullback diagram below shows that the result follows as claimed.



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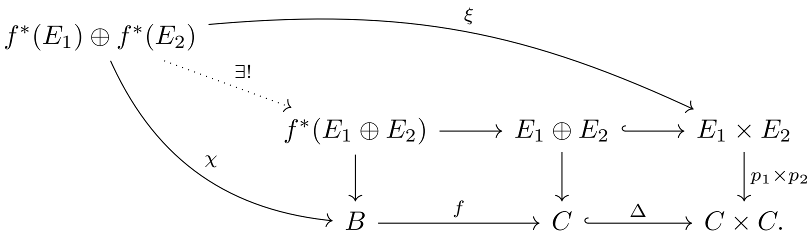
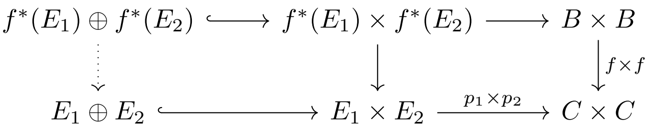
Proof. (3) By (1), it is sufficient to show that there is an isomorphism:

$$f^*(E_1) \oplus f^*(E_2) \approx (\Delta f)^*(E_1 \times E_2) \tag{E1}$$

fitting in the diagram:

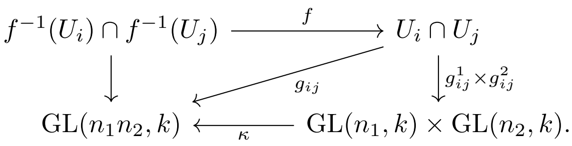


χ is defined by $p_1^{-1}(f(b)) \times p_2^{-1}(f(b)) \ni (v_1^b, v_2^b) \mapsto b \in B$ and ξ a canonical composition $f^*(E_1) \oplus f^*(E_2) \hookrightarrow f^*(E_1) \times f^*(E_2) \to E_1 \times E_2$, where the first map of ξ can be replaced by the induced map to $E_1 \oplus E_2$, by the pullback cone $C \xrightarrow{\Delta} C \times C \xleftarrow{p_1 \times p_2} E_1 \times E_2$ as such.



ξ and χ gives rise to the isomorphism (E1) given by $(v_1^b, v_2^b) \mapsto (\chi(v_1^b), \xi(v_1^b, v_2^b)) = (b, v_1', v_2')$, which is a fiber-wise linear isomorphic continuous map of bundles over B. ∎

Proof. (4) $f^*(E_1 \otimes E_2)$ and $f^*(E_1) \otimes f^*(E_2)$ are safely assumed to have the same open covering $\{f^{-1}(U_i)\}_i$ of the base space B with respect to which E_1 and E_2 both locally trivialize over $U_i \subset C$. Then according to the gluing construction of a vector bundle over a Čech cocycle, we only need to check the transition functions yield the same actions on the model fibers. Indeed, the following diagram commutes by definition of g_{ij}, where κ is the induced bilinear function of k-algebra:



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Theorem 1. Given a vector bundle $p : E \to B$ and homotopic maps $f_0, f_1 : A \to B$, then the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is compact Hausdorff or more generally paracompact.

By the factorization $A \times \{i\} \hookrightarrow A \times I \xrightarrow{F} B$ of f_i $(i = 0, 1)$ through a homotopy $F : A \times I \to B$ from f_0 to f_1, it is sufficient to show that the restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.

As the first step, let us show the two preliminary facts to use in the proof of Theorem 1.

(P1) A vector bundle $p : E \to X \times [a, b]$ is trivial if its restrictions over $X \times [a, c]$ and $X \times [c, b]$ are both trivial for some $c \in (a, b)$.

(P2) For a vector bundle $p : E \to X \times I$, there exists an open cover $\{U_\alpha\}$ of X so that each restriction $p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial.

Proof. (P1) Assume we are given isomorphisms $h_1 : E_1 = p^{-1}(X \times [a, c]) \to X \times [a, c] \times k^n$ and $h_2 : E_2 = p^{-1}(X \times [c, b]) \to X \times [c, b] \times k^n$. By replacing h_2 with the composition:

$$E_2 \xrightarrow{h_2} X \times [c, b] \times k^n \xrightarrow{c} X \times \{c\} \times k^n \xrightarrow{h_1 h_2^{-1}} X \times \{c\} \times k^n \hookrightarrow X \times [c, b] \times k^n,$$

we get the isomorphism $h_1 \cup h_2 : E \to X \times [a, b] \times k^n$ that (globally) trivializes E. □

Proof. (P2) For each $x \in X$, we can find open neighbourhoods $U_{x,1}, \ldots, U_{x,k}$ in X and a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0, 1]$ such that the bundle is trivial over $U_{x,i} \times [t_{i-1}, t_i]$, using compactness of $[0, 1]$. Then by (P1) the bundle is trivial over $U_\alpha \times I$ where $U_\alpha = U_{x,1} \cap \ldots \cap U_{x,k}$. □

Proof. (**Theorem 1**) As in the book [1] shows, the essential part of the proof resides in the finite case, hence we will write only when B is compact Hausdorff.

Let $\{U_\alpha\}$ be an open cover of X such that $p^{-1}(U_\alpha \times I)$ is a trivial bundle (by (P2), these cover always exists). We can assume $\mathcal{U} = \{U_\alpha\} = \{U_1, \ldots, U_m\}$ since X is assumed to be compact.

Let $\{u_i\}_{i \le m}$ be a partition of unity subordinate to $\mathcal{U}$ and define a function $\xi_k : X \to I$ by

$$\xi_k = u_1 + u_2 + \ldots + u_k \quad (0 \le k \le m).$$

Notice in particular $\xi_0 = 0$ and $\xi_m = 1$ on X.

A series of graphs $\Gamma_{\xi_k} \subset X \times I$ of ξ_k yield the restricted bundles $p_k : E_k \to \Gamma_{\xi_k}$ that fit into the following commutative diagram:

$$\begin{array}{ccccccccc}
E_0 & \xrightarrow{h_1} & E_1 & \xrightarrow{h_2} & \ldots & \xrightarrow{h_{m-1}} & E_{m-1} & \xrightarrow{h_m} & E_m \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & & & \downarrow{\scriptstyle p_{m-1}} & & \downarrow{\scriptstyle p_m} \\
X \times \{0\} = \Gamma_{\xi_0} & \longrightarrow & \Gamma_{\xi_1} & \longrightarrow & \ldots & \longrightarrow & \Gamma_{\xi_{m-1}} & \longrightarrow & \Gamma_{\xi_m} = X \times \{1\},
\end{array}$$

where the bottom arrows are homeomorphisms (of graphs to the domain X).

To construct the lifts (dotted arrows), note that $\xi_k = \xi_{k-1}$ on $x \in X \backslash U_k$ and hence $E_k = E_{k-1}$ over $(X \backslash U_k) \times I$. Because E is trivialized over $U_k \times I$, so does over the restriction to each graphs that amount to the union of homeomorphisms:

$$\begin{array}{ccccc}
p_k^{-1}((X \backslash U_k) \times I) & \hookrightarrow & E_k & \hookleftarrow & p_k^{-1}(U_k \times I) \approx U_k \times I \times \mathbb{R}^n \\
\| & & \uparrow{\scriptstyle \exists h_k} & & \uparrow{\scriptstyle \approx} \\
p_{k-1}^{-1}((X \backslash U_k) \times I) & \hookrightarrow & E_{k-1} & \hookleftarrow & p_{k-1}^{-1}(U_k \times I) \approx U_k \times I \times \mathbb{R}^n.
\end{array}$$

h_k are trivially isomorphisms and the composition $h_m \circ \cdots \circ h_1$ is the desired bundle isomorphism. □

References

[1] A. Hatcher. *Vector Bundles and K-Theory.* http://www.math.cornell.edu/~hatcher. 2003.