Note on (p, q)-shuffles and its canonical order

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Notation. For $p \geq 1$, we denote by (p) the totally ordered set $(p) = \{1 < 2 < \ldots < p\}$ of p elements, and $[p]$ the totally ordered set $[p] = \{0 < 1 < \ldots < p\}$ of $p + 1$ elements.

Notation. For any set $A \in \mathbf{Set}$, we denote by $\mathcal{S}(A)$ the symmetric (a.k.a. permutation) group of A.

Notation. We denote by **Ord** the category of partially ordered (pre-ordered) set as an object with the order preserving functions between them as the morphisms (it doesn't change any of the later arguments when the objects are taken to be a set of pre-ordered sets).

Definition. For $p, q \geq 1$, the *(p,q)-shuffle* $\mathrm{Shuffle}(p, q)$ is a subset of $\mathcal{S}((p+q))$ defined with the additional monotone property:

$$\mathrm{Shuffle}(p, q) = \left\{\sigma \in \mathcal{S}((p + q)) : \sigma(1) < \sigma(2) < \ldots < \sigma(p) \wedge \sigma(p + 1) < \sigma(p + 2) < \ldots < \sigma(p + q)\right\}.$$

Notation. For $p, q \geq 1$, we denote by $\mathrm{Shuffle}_s(p, q)$ the set of all surjective monotone functions (surjective morphisms) $(p) \sqcup (q) \twoheadrightarrow (p + q)$ on **Ord**, where the coproduct is taken in the category.

Notation. For $p, q \geq 0$, we denote by $\mathrm{Shuffle}_i(p, q)$ the set of all injective monotone functions (injective morphisms) $[p + q] \hookrightarrow [p] \times [q]$ on **Ord**, where the product is taken in the category.

Note. $\mathrm{Shuffle}_s(p, q)$ has a structure of the partial order defined by

$$\sigma \leq \sigma' \iff \sigma|_p(k) \leq \sigma'|_p(k) \text{ for all } 1 \leq k \leq p.$$

Proposition 1. There are bijections:

$$\mathrm{Shuffle}(p, q) \approx \mathrm{Shuffle}_s(p, q) \approx \mathrm{Shuffle}_i(p, q).$$

Proof. **Proof of** $\mathrm{Shuffle}(p, q) \approx \mathrm{Shuffle}_s(p, q)$

For $\sigma_s \in \mathrm{Shuffle}_s(p, q)$, the cardinalities of domain and codomain imply that σ_s indeed is a bijection with the monotone property; conversely, each element of $\mathrm{Shuffle}(p, q)$ defines the surjective monotone function on the coproduct $(p) \sqcup (q)$ by definition.

Proof of $\mathrm{Shuffle}_s(p, q) \approx \mathrm{Shuffle}_i(p, q)$

For $p, q \geq 1$ and $\sigma_s \in \mathrm{Shuffle}_s(p, q)$, define σ_i by

$$\sigma_i(l) = \begin{cases} (0, 0) & if\ l = 0, \\ \sigma_i(l - 1) + (\delta_p(l), \delta_q(l)) & if\ 1 \leq l \leq p + q, \end{cases}$$

where

$$\delta_r(l) = \begin{cases} 1 & if\ \sigma_s^{-1}(l) \in (r), \\ 0 & otherwise. \end{cases}$$

for $r = p, q$.

By definition, σ_i is an injective monotone function to the product order $[p] \times [q]$ such a way that σ_i is represented by an ordered sequence of pairs such as $\langle(0, 0), (1, 0), (1, 1), (2, 1), \ldots, (p, q)\rangle$, beginning from $(0, 0)$, incrementing by one on either factor and ending to (p, q).

Let $\sigma_i \in \mathrm{Shuffle}_i(p,q)$ for $p,q \geq 1$. By definition of coproduct (in **Ord**), we can define σ_s by a pair of monotone functions $\sigma_s|_p : (p) \to (p+q)$ and $\sigma_s|_q : (q) \to (p+q)$ that sum up to the induced surjective function σ_s.

Define $\sigma_s|_p$ by

$$\sigma_s|_p(l) = \sigma_i^{-1}(l, m_l)$$

for some $l \in (p)$ and $m_l \in [q]$ such that

$$\sigma_i^{-1}(l, m_l) = \sigma_i^{-1}(l-1, m_l) + 1. \tag{1}$$

This is well-defined since for fixed $l \in (p)$, if (l,m) suffices (1) for some $m \in [q]$, then no $m' \in [q]$ other than m has non-empty inverse images $\sigma_i^{-1}(l,m')$ and $\sigma_i^{-1}(l-1,m')$, by assumption of σ_i being injective monotone.

By defining $\sigma_s|_q$ in the similar manner, we have the induced function

$$\sigma_s = \sigma_s|_p \sqcup \sigma_s|_q : (p) \sqcup (q) \to (p+q).$$

This is surjective since for each $k \in (p+q)$, there is $l \in (p)$ such that $\sigma_s|_p(l) = k$ when $k \leq p$, and there is $l' \in (q)$ such that $\sigma_s|_q(l') = k$ when $p+1 \leq k$. □

Corollary 1. The equivalences of **Proposition 1** in **Set** can be made into the equivalences in **Ord**.

Proof. Observing that an element of $\mathrm{Shuffle}(p,q)$ is completely determined by the choice of order-insensitive p elements out of $p+q$ elements, each (p,q)-shuffle $\sigma \in \mathrm{Shuffle}(p,q)$ is represented by an ordered tuple $\langle \sigma^1, \sigma^2, \dots, \sigma^p \rangle$ with $\sigma^k \in (p+q)$ for $1 \leq k \leq p$, it is clear that $\mathrm{Shuffle}(p,q)$ yields the obvious partial order through the bijection compatible with that of $\mathrm{Shuffle}_s(p,q)$, namely $\sigma \leq \tau$ iff $\sigma^k \leq \tau^k$ for all k.

The corresponding order on $\mathrm{Shuffle}_i(p,q)$ is analogously defined by $\sigma_i \leq \tau_i$ iff $\sigma_i^{(q,k)} \leq \tau_i^{(q,k)}$ for all $k \in [p+q]$, when we represent σ_i by the (truncated) $(p+q-1)$ tuple of pairs,

$$\sigma_i = \langle (\sigma_i^{(p,1)}, \sigma_i^{(q,1)}), (\sigma_i^{(p,2)}, \sigma_i^{(q,2)}), \dots, (\sigma_i^{(p,p+q-1)}, \sigma_i^{(q,p+q-1)}) \rangle,$$

where $(\mathrm{prj}_r \circ \sigma_i)(k) = \sigma_i^{(r,k)}$, $\sigma_i^{(r,0)} = 0$ and $\sigma_i^{(r,p+q)} = r$ for $r = p,q$.

This indeed yields an equivalence in **Ord** as in the following equations:

$$\begin{array}{ll}
\sigma_s \leq \tau_s \text{ in } \mathrm{Shuffle}_s(p,q) & \Longleftrightarrow \\
\sigma_s^k \leq \tau_s^k,\ \forall k \in (p+q) & \Longleftrightarrow \\
\delta_q(\sigma_s^k) \leq \delta_q(\tau_s^k),\ \forall k \in (p+q) & \Longleftrightarrow \\
\sigma_i^{(q,k)} \leq \tau_i^{(q,k)},\ \forall k \in [p+q] & \Longleftrightarrow \\
\sigma_i \leq \tau_i \text{ in } \mathrm{Shuffle}_i(p,q). &
\end{array}$$

□

References

[1] Kerodon. *The Shuffle Product*. https://kerodon.net/tag/00RF. Accessed: (Dec. 28). 2023.