K-group and its fundamental product theorem

stma

January 28, 2024

2 Vector Bundle

2.8 Orientation

Definition. A *frame* of n-vector space $V \approx k^n$ is an n-tuple of ordered basis $\langle v_1, \dots, v_n \rangle$.

Definition. The *frame bundle* of a given n-vector bundle $p : E \to B$ is an associated fiber bundle $F(E) \to B$ equipped with a set of frames as the model fiber.

Lemma 1. For a given n-vector bundle $p : E \to B$, the frame bundle $Fp : F(E) \to B$ is a fiber bundle.

Proof. As a set, a set of frames of the model fiber k^n of E is identified with the general linear group of degree n, $\mathrm{GL}(n, k)$, hence we denote $Fp^{-1}(x) = \mathrm{GL}(n, k)$. Furthermore, by identifying the model fiber k^n with the diagonal matrix $\mathrm{diag}(a_1, \dots, a_n) \in \mathrm{GL}(n, k)$, the frame bundle construction is viewed as a sort of fiber extension in a sense that (the image of) a transition map $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, k)$ of E is interpreted as a special case of left action

$$\begin{array}{ccc} k^n & = & k^n \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle g\gamma} \\ k^n & \xrightarrow{g} & k^n, \end{array}$$

where $\gamma = 1_{k^n}$ and $g = g_{ij}(x)$ in case of E. This shows that the local trivialization of $F(E)$ over $U \subset B$ is given as the extension in **Top**

$$\begin{array}{ccc} U \times k^n & \xrightarrow{h} & p^{-1}(U) \\ \downarrow & & \downarrow \\ U \times \mathrm{GL}(n, k) & \xrightarrow{Fh} & Fp^{-1}(U). \end{array}$$

□

The transition map $U_i \cap U_j \to \mathrm{Aut}(\mathrm{GL}(n, k))$ over (i, j)-coordinates of $F(E)$ is completely represented by the left action of topological group $\mathrm{GL}(n, k)$. This is because the action is free and transitive, making $F(E)$ so-called *principle* $\mathrm{GL}(n, k)$*-bundle*. In particular, $\mathrm{Aut}(\mathrm{GL}(n, k))$ can be replaced by $\mathrm{GL}(n, k)$.

Definition. A n-vector bundle $p : E \to B$ is called *orientable* if it admits an (ordered) n-tuple of local sections $s = \langle s_1, \dots, s_n \rangle$ (i.e. a cocone $s : U \dot{\to} \mathrm{GL}(n, k)$ in **Top** where $U : \Lambda \to$ **Top** is the assignment of coordinate charts from the set of indices) that is linearly independent in each fiber and suffices the following locally consistent property:

For any indices α, β of coordinate charts with $U_\alpha \cap U_\beta \neq \varnothing$, the diagram

$$U_\alpha \cap U_\beta \underset{\langle s_1 \dots s_n \rangle_\alpha}{\overset{\langle s_1 \dots s_n \rangle_\beta}{\rightrightarrows}} \mathrm{GL}(n, k) \xrightarrow{\mathrm{sgn}} \{\pm 1\} \qquad \text{(LCP)}$$

is a cofork.

Note. The orientability of a vector bundle can be defined by the existence of a family of local sections with (LCP), to the associated frame bundle.

2.9 Universal Bundle

For fixed $n \geq 0$, there is the *universal vector bundle* of dimension n, denoted by $E_n \to G_n$, by which every n-dimensional vector bundle over a chosen paracompact space X is obtained as the pullback of $X \to G_n \leftarrow E_n$.

Fact. For paracompact space X, the map $[X, G_n] \to \mathrm{Vect}_n(X)$, $[f] \mapsto f^*(E_n)$, is a bijection [1, Theorem 1.16].

Note. There are several variants of above result. For instance the analogous result holds over the complex number field instead of the real number field, by setting G_n the spaces of n-dimensional $\mathbb{C}$-linear subspaces $G_n(\mathbb{C})$. There are orientable versions, etc.

3 K-group

3.1 Group completion of a commutative monoid

There is the left adjoint to the forgetful functor $U : \mathbf{Ab} \to \mathbf{CMon}$, the free functor F, which gives rise to the existence of the canonical abelian group from a given commutative monoid, in the form of adjoint unit; whereas the fact does not allude any explicit construction of such abelian group, which is presented next.

Let M be a commutative monoid. Consider the canonical (commutative) monoid structure of the product $M \times M$, on which the equivalence relation $\sim$ is defined by

$$(m_1, m_2) \sim (n_1, n_2) \iff \exists k \in M, m_1 + n_2 + k \approx n_1 + m_2 + k,$$

where the equivalence $\approx$ is taken in M.

The quotient set $K(M) = (M \times M)/\sim$, for which we denote the equivalence class of (m_1, m_2) by $[m_1, m_2]$, yields the (abelian) group structure with the inverse element $[m_2, m_1]$.

As the fractions analogously characterizes the equivalence and cancellation property employed in $K(M)$, it is justifiable to write $m_1 - m_2$ for the place of $[m_1, m_2]$. In particular, $m_1 - m_2 \approx m_1 + n - m_2 + n$ holds for arbitrary $n \in M$ in general.

3.2 K-group of vector bundles

By restricting the class of concerned base spaces to the compact Hausdorff spaces, each vector bundle $E \to B$ admits the stable inverse, meaning that there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is the trivial bundle [1, Proposition 1.4].

This property is required for defining reduced K-group, to be compatible with the relative theory that gives rise to the connection with the generalized cohomology theory; furthermore, Bott periodicity. So we assume that the base spaces are compact Hausdorff from here on, otherwise stated.

Moreover, we start investigating the objects with the fiber isomorphic to a vector space over complex number field, of locally constant dimensions. This assumption contains even dimensional real vector bundles for the restriction $\mathrm{Vect}^n_{\mathbb{C}}(X) \to \mathrm{Vect}^{2n}_{\mathbb{R}}(X)$ of the fiber-wise multiplications being surjective, with regard to the canonical complex structure on $\mathbb{R}^{2n}$ (i.e. $J(x_1, \ldots, x_{2n}) = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1})$). We mean by the fiber has *locally constant dimensions* that each fiber may differ with the dimension of isomorphic vector space.

With these in mind, we are introducing the construction of *K-group* (actually a ring) over a chosen compact Hausdorff space and its basic properties in the following line.

1. Ring construction → 2. Fundamental product theorem → 3. Bott periodicity

In this text, we are dealing with the first two out of three (blue boxes).

3.2.1 Ring construction

Definition. On a set of isomorphism classes of vector bundles (with locally constant dimensional fibers) Vect(B) over B, we denote *stable isomorphic* bundles by $E \approx_s E'$ meaning that there exists some $n \in \mathbb{Z}$ such that $E \oplus \epsilon^n \approx E' \oplus \epsilon^n$. We denote *weakly stably isomorphic* bundles by $E \sim E'$ meaning that there exists some $n, m \in \mathbb{Z}$ such that $E \oplus \epsilon^n \approx E' \oplus \epsilon^m$.

Note. The stable isomorphism and weakly stable isomorphism are equivalence relations on Vect(B), where the stable isomorphism is strictly stronger notion.

Fact. Over a compact Hausdorff space B, the stable isomorphism classes in Vect(B) yield commutative monoid structure $\langle \oplus, \epsilon^0 \rangle$, while the weakly stable isomorphism classes, denoted by $\tilde{K}(B)$, yields an abelian group structure $\langle \oplus, \epsilon^0 \rangle$.

Proposition 1. Over a compact Hausdorff space B, the group completion of the stable isomorphism classes in Vect(B) yields a commutative ring structure $\langle +, [E - E], \cdot, [\epsilon^1 - \epsilon^0] \rangle$, denoted by $K(B)$, with a ring isomorphism

$$K(B) \cong \tilde{K}(B) \oplus \mathbb{Z}.$$

Furthermore, $\tilde{K}(B)$ admits a commutative (non-unital) ring structure $\langle \oplus, [E], \cdot \rangle$ with the induced multiplication from $K(B)$.

Proof. The operations are defined as followings on $K(B)$.

(addition)

$$[E - E'] + [F - F'] \quad = \quad [E \oplus F - E' \oplus F'];$$

(multiplication)

$$\begin{aligned}[E - E'] \cdot [F - F'] &= [E \otimes F - E' \otimes F] + [E' \otimes F' - E \otimes F'] \\ &= [E \otimes F - E \otimes F'] + [E' \otimes F' - E' \otimes F].\end{aligned}$$

In the definition of the multiplication, the second and the third term are equivalent. The operations are both well-defined. While we are not giving here full expositions with regard to ring properties, some of them are shown as followings.

(additive unit) $[E - E'] + [E'' - E''] = [E \oplus E'' - E' \oplus E''] = [E - E'];$

(multiplicative unit) $[E - E'] \cdot [\epsilon^1 - \epsilon^0] = [E \otimes \epsilon^1 - E' \otimes \epsilon^1] + [E' \otimes \epsilon^0 - E \otimes \epsilon^0] = [E - E'];$

(distributive) $[E_1 - E_1'] \cdot ([E_2 - E_2'] + [E_3 - E_3']) = [E_1 - E_1'] \cdot [E_2 - E_2'] + [E_1 - E_1'] \cdot [E_3 - E_3'].$

To see the ring isomorphism, observe that the function

$$\psi : K(B) \to \tilde{K}(B); \quad [E - \epsilon^m] \mapsto [E];$$

is a surjective ring homomorphism with the kernel of the form $[\epsilon^n - \epsilon^m]$, which is enumerated by $n - m$, isomorphic to $\mathbb{Z}$. There is the left inverse of $\mathbb{Z} \ni n \mapsto [\epsilon^n - \epsilon^0] \in K(X)$ given by a restriction $[E - \epsilon^m] \mapsto [E|_{b_0} - \epsilon^m|_{b_0}] \in K(b_0)$ to some point $b_0 \in B$. The function defined by

$$\eta : \tilde{K}(B) \to K(B); \quad [E] \mapsto [E - E|_{b_0}];$$

is a right inverse of ψ, for a fixed $b_0 \in B$, hence we have a split exact sequence:

$$\mathbb{Z} \approx K(b_0) \leftrightarrows K(B) \underset{\psi}{\overset{\eta}{\leftrightarrows}} \tilde{K}(B),$$

depending on the choice of b_0.

The multiplication in $\tilde{K}(B)$ is defined with respect to the following diagram:

$$\begin{array}{ccc} K(B) \otimes K(B) & \xrightarrow{\ \cdot\ } & K(B) \\ {\scriptstyle \eta\otimes\eta}\uparrow & & \downarrow{\scriptstyle \psi} \\ \tilde{K}(B) \otimes \tilde{K}(B) & \overset{\cdot}{\dashrightarrow} & \tilde{K}(B). \end{array}$$

To make the diagram commute (up to the equivalence), it is explicitly defined by:

$$[E] \cdot [E'] = [EE' \oplus \overline{E|_{b_0}E' \oplus E'|_{b_0}E}],$$

where $\overline{E}$ denotes a stable inverse of E.

For each map $f : X \to Y$, the induced map $f^* : \tilde{K}(Y) \to \tilde{K}(X)$ is a ring homomorphism since

1. $f^*(E \oplus E') = f^*(E) \oplus f^*(E')$ up to isomorphism;
2. $f^*(E \otimes E') = f^*(E) \otimes f^*(E')$ up to isomorphism; (PBP)
3. $f^*(\overline{E}) = \overline{f^*(E)}$ up to weakly stable isomorphism.

hold.

□

Note. Above exact sequence implies that $K(B)$ decomposes into the two sub-modules (ideals) $\mathbb{Z}$ of unital component and $\tilde{K}(B)$ of non-unital one, which account for a "stably trivial dimension" (of the negative factor) and a "stably non-trivial twist" of the class of bundles, respectively.

3.2.2 External product

An *external product* can be defined by

$$\mu : K(X) \otimes K(Y) \to K(X \times Y); \quad a \otimes b \mapsto p_X^*(a) \cdot p_Y^*(b).$$

Lemma 2. μ is a ring homomorphism.

Proof. The basic properties (PBP) of pullback induced maps immediately deduce that μ is a homomorphism of an abelian group. As a map from a ring of tensor product of rings, it holds that

$$\begin{aligned}
\mu((a \otimes b)(a' \otimes b')) &= \mu(aa' \otimes bb') \\
&= p_X^*(aa') \cdot p_Y^*(bb') \\
&= p_X^*(a) \cdot p_Y^*(b) \cdot p_X^*(a') \cdot p_Y^*(b') \\
&= \mu(a \otimes b) \cdot \mu(a' \otimes b').
\end{aligned}$$

□

In a homotopy theoretic argument of the canonical line bundle $H \to \mathbb{C}P^1 \cong S^2$ [1, Example 1.13], the homotoped clutching functions $f, g : S^{k-1} \to \mathrm{GL}_{2n}(\mathbb{C})$ yield the bundle isomorphism

$$(H \otimes H) \oplus \epsilon^1 \approx H \oplus H,$$

which is interpreted as $(H-1)^2 = 0$ in the image of ring homomorphism $\mathbb{Z}[H] \to K(S^2)$, inducing a ring homomorphism $\lambda : \mathbb{Z}[H]/(H-1)^2 \to K(S^2)$.

Fact. *(The fundamental product theorem of K-group)*
The composition

$$\xi : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{1 \otimes \lambda} K(X) \otimes K(S^2) \xrightarrow{\mu} K(X \times S^2)$$

is an isomorphism of rings for all compact Hausdorff spaces X [1, Theorem 2.2].

References

[1] A. Hatcher. *Vector Bundles and K-Theory.* http://www.math.cornell.edu/~hatcher. 2003.