Note on Laplace transform and its uniqueness

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Definition. A function $f : X \to Y$ from a subspace of $\mathbb{R}$ is called *piecewise continuous* if there exists a family of disjoint open intervals $\{V_n\}_n$ such that f is continuous on V_n for all n and $X - \bigcup_n V_n$ is a discrete subspace of X.

Note. Because $\mathbb{R}$ is second-countable, the same holds for every subspaces, hence the difference $\mathbb{R} - \bigcup_n V_n$, on which f may be discontinuous, is discrete countable.

Lemma 1. A bounded discrete subspace D of $\mathbb{R}$ is a finite set.

Proof. Since D is countable, we can assume D is a sequence of real numbers $\{x_n\}_{n\geq 1}$ with $x_n \neq x_m$ if $n \neq m$.

Let us define a subset of D by

$$D(n) = \{x_m \in D | 1 \leq m < n\}.$$

If D is infinite, the maximal distance to a subset $d_n = \inf_{k\leq n} \sup_{y\in D(k)} |x_k - y| \leq A$ is decreasing that limits to zero when $n \to \infty$. This implies that D contains some accumulation point, contradicts to D being discrete. □

Corollary 1. A piecewise continuous function $f : B \to Y$ from a bounded domain $B \subset \mathbb{R}$ admits utmost finite number of discontinuous points.

Proof. Trivial. □

Definition. A function $f : [v, \infty) \to \mathbb{R}$ is called *of exponential order* if there exists $c, A \in \mathbb{R}$ and $M \in \mathbb{R}_{\geq 0}$ such that $|f(t)/e^{ct}| \leq M$ for all $t > A$.

Definition. A Laplace transform $\mathcal{L}f$ of a function $f : [0, \infty) \to \mathbb{R}$ is the integral defined by

$$\mathcal{L}f = \int_0^\infty e^{-st} f(t) dt.$$

Note. The Laplace transform $\mathcal{L}f(s)$ is a function of $s \in U$ for some open set $U \subset \mathbb{R}$ if the integral converges absolutely.

Proposition 1. A piecewise continuous function $f : [0, \infty) \to \mathbb{R}$ of exponential order admits a Laplace transform $\mathcal{L}f(s)$ on $s > c$ for some $c \in \mathbb{R}$.

Proof. By assumption, there exists $c, A \in \mathbb{R}$ and $M \in \mathbb{R}_{\geq 0}$ such that the following inequalities hold.

$$\begin{aligned}
|\mathcal{L}f| &\leq \int_0^\infty |e^{-st} f(t)| dt \\
&\leq \int_0^A e^{-st} |f(t)| dt + M \int_A^\infty e^{-(s-c)t} dt \\
&< \infty \quad (s > c).
\end{aligned}$$

The first integral is finite because the continuous integrand $|e^{-st} f(t)|$ yields finite integral on each subinterval, amounts to a finite value over the finite number of subintervals of $[0, A]$, by Corollary 1. The second integral is calculated to be $\frac{M}{s-c} e^{(s-c)A}$. □

Fact 1. (Weierstrass approximation theorem) Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial p such that for all $x \in [a, b]$, we have $|f(x) - p(x)| < \epsilon$.

Theorem 1. (Uniqueness of Laplace transform) Let $f, g : [0, \infty) \to \mathbb{R}$ be two piecewise continuous functions of exponential order such that $\mathcal{L}f(s) = \mathcal{L}g(s)$ for all $s > c$. Then $f(t) = g(t)$ at all t where both are continuous.

Proof. Due to the linearity of Laplace transform, namely $\mathcal{L}f(s) - \mathcal{L}g(s) = \mathcal{L}(f - g)(s)$, it suffices to show that $f(t) \equiv 0$ if $\mathcal{L}f(s) \equiv 0$. Furthermore, we put $s = s_0 + n + 1$ for some fixed $s_0 > c$ (we don't need to exploit all possible $s > c$ in order to deduce $f(t) \equiv 0$ at all t where f is continuous) and $n \in \mathbb{Z}_{\geq 0}$ that we have:

$$\begin{aligned}\mathcal{L}f(s) &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^1 u^n u^{s_0} f(-\log(u)) du \qquad (u = e^{-t}, n \in \mathbb{Z}_{\geq 0}) \\ &= 0\end{aligned} \tag{1}$$

Put

$$\beta(u) = u^{s_0} f(-\log(u)) \qquad (0 \leq u \leq 1).$$

Then,

$$\int_0^1 u^n \beta(u) du = 0 \qquad (n \in \mathbb{Z}_{\geq 0}), \tag{2}$$

and it turns out that $\beta(1) = f(0)$ by virtue of f being continuous around 0, and

$$\lim_{u \to +0} \beta(u) = \lim_{t \to \infty} e^{-s_0 t} f(t) = \lim_{t \to \infty} O(e^{-(s_0 - c)t}) = 0,$$

hence β is piecewise continuous on $[0, 1]$.

Applying Fact 1, for arbitrary $\epsilon_j > 0$, we have a sequence of polynomials $p_1, \ldots, p_m$ such that

$$\|\beta|_{V'_j} - p_j\| < \epsilon_j,\ 1 \leq j \leq m,$$

where $\{V'_j\}$ are disjoint open sets of $[0, 1]$ on which β is continuous.

By setting $p = \sqcup_j p_j : [0, 1] \to \mathbb{R}$, assigning $p(d) = 0$ to each discontinuous point $d \in (0, 1)$ if necessary, we have $\int_0^1 p(u)\beta(u) du = 0$ from (2) and the linearity of integral.

Put $\epsilon = \sum \epsilon_j$, it follows that

$$\begin{aligned}\int_0^1 |\beta(u)|^2 du &= \int_0^1 |\beta(u)(\beta(u) - p(u))| du \\ &= \sum_{j=1}^m \int_{V'_j} |\beta(u)(\beta(u) - p_j(u))| du \\ &\leq \epsilon \int_0^1 |\beta(u)| du.\end{aligned} \tag{3}$$

Because ϵ is arbitrary,

$$|\beta(u)|^2 = \int_0^u |\beta(v)|^2 dv \leq \int_0^1 |\beta(u)|^2 du \equiv 0$$

holds for any continuous point $u \in [0, 1]$, the claim follows. □

References

[1] David Vernon Widder. *Laplace transform (PMS-6)*. Princeton university press, 2015.