Note on the element generation of an algebraic colimit

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While a colimit is of a typical object in category theory hence so much in the closely related areas such as categorical logic and type theory, its construction can be a subtle issue especially within a strict category setting where elementhood matters in the objects.

We describe a concrete method of "generating an element of colimit" in terms of so called solution set as a part of necessary and sufficient conditions of adjoint functor theorem. The main ideas are originated from [3].

1 The solution set condition and its interpretations

The *solution set condition*, originally formalized as a necessary condition that a functor (under some condition) yields a left adjoint, is an interesting assertion by itself simply because it gives an explicit construction of a colimit.

More precisely, *solution set condition* denotes the existence of "weakly universal objects" in some sense that can be expressed (or unified) in terms of functor among the following (general) *Adjoint functor theorem* and its corollaries:

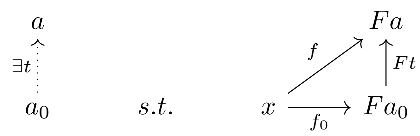
- Adjoint functor theorem;
- Condition for the existence of initial object;
- Representable functor theorem.

Let A be a locally small, complete small category and let F denote a functor from the specified domain A, where the codomain is the key feature that characterizes what the solution set is for.

For given functor $F : A \to X$, the (unified) *solution set condition* is concisely stated as:

$$\forall x \in Ob(X), \mathrm{wInit}(x \downarrow F) \neq \varnothing \quad (1)$$

where wInit denotes a (small) set of *weakly initial objects*, whose element suffices the condition of initial object except the uniqueness condition. (1) may be seen clearly in the diagram



The condition (1) will be interpreted accordingly in a context.

1.1 Condition for the existence of initial object

When $F : A \to \mathbf{1}$ is a constant functor, we have a necessary *condition for the existence of initial object*, i.e.

$$\mathrm{wInit}(1 \downarrow F) \neq \varnothing.$$

by setting $X = \mathbf{1}$ in the previous notation. Note that F canonically preserves every (small) limits because any universal cone $\tau : a \dot{\to} P \in A^J$ collapses to the zero object in $\mathbf{1}$.

1.2 Representable functor theorem

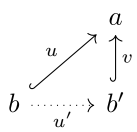
Analogous to the *condition for the existence of initial object*, a representation of $F : A \to X$ is identified with the universal arrow $\mathrm{Init}(* \downarrow F)$ for $X = \mathbf{Set}$, or equivalently the *unit of adjoint* $\eta : I_{\mathbf{1}} \dot{\to} F'S$, where $\mathbf{1}$ is identified with the faithful subcategory $* \subset X = \mathbf{Set}$, F' the composition of F followed by $X \to \mathbf{1}$ and $S : \mathbf{1} \to A$ a choice functor.

The conclusion is remarkable to restate.

Fact. F is representable if and only if $\mathrm{wInit}(* \downarrow F) \neq \varnothing$ and F preserves small limit.

2 Subobjects and spanning maps

For an arbitrary category A, we can consider "subobjects" of an object $a \in A$, whose elements are monos $u : b \to a$ together with the order relation defined by $u \leq v$ if and only if there exists some $u' : b \to b'$ and a mono $v : b' \to a$ such that $u = v \circ u'$, or to say concisely u factors through v.



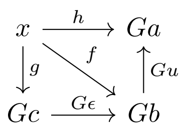
By defining $(u \leq v) \wedge (v \leq u) \iff u = v$, we call the set of equivalence classes *subobjects of* a. We denote subobjects of a by $Sub(a) \in \mathbf{Pos}$.

Definition. For a given functor $G : A \to X$, a map $f : x \to Ga$ is said to **span** a when there is no (non-isomorphic) mono $s \to a$ such that f factors through $Gs \to Ga$.

Lemma 1. Fix a category A and its object $a \in A$. Suppose the pullbacks for any set of subobjects $S \subset Sub(a)$ exists, denoted by $\cap S \in Sub(a)$. If a functor $G : A \to X$ preserves the all such pullbacks, then every map $h : x \to Ga$ factors through a map $f : x \to Gb$ that spans b.

Proof. For $h : x \to Ga$, let $U = \{u_i : b_i \to a\} \subset Sub(a)$ be a set of subobjects such that $\exists f_i : x \to Gb_i$ with $h = Gu_i \circ f_i$ for each i. Since by assumption there exist the pullback $\cap U \in Sub(a)$ which we denote as $u : b \to a$, we see it suffices $h = Gu \circ f$, where f is the induced map on pullback diagrams of $\{Gb_i \xrightarrow{Gu_i} Ga \xleftarrow{Gu} Gb\}_i$.

f apparently spans b by the construction of U; precisely, if there is a mono $(\epsilon : c \to b) \in Sub(b)$ such that f factors through $G\epsilon$ for some $g : x \to Gc$ (i.e. $f = G\epsilon \circ g$), then $u \circ \epsilon \in Sub(a)$ must be in U and since u is the pullback, c must coincide with b, analogously g with f and ϵ with id_b.



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3 Universal Algebra

There is a way of modelling vast variety of algebras in a unified manner, called *universal algebra*, which assigns a type of algebraic system to a given set. While the original formulation leads back to 1960's, when the works by Cohn [1] and Grätzer [2] were published, the idea is not faded and worth noting here.

Definition. The *type* τ of algebraic system is a pair $\tau = \langle \Omega, E \rangle$ of a graded set of operators Ω and a set of identities E.

An operator $\omega \in \Omega$ is endowed with the number of parameters $n(\omega)$ called *arity*, together with a given *underlying set* S, we have the *action of* Ω *on* S defined by

$$\omega \mapsto (\omega_{\mathcal{A}} : S^{n(\omega)} \to S)$$

where $\mathcal{A}$ is a realized set of algebra (equivalently, an algebraic system) associated with the type τ, or τ-algebra.

Although the set of operators are to be extended by definition of τ-algebra, uniquely to that of derived operators Λ by *assignment* and *composition* of operators, we identify Ω with a set of derived operators, Λ. Here we remark the uniqueness of the extension.

$$\begin{array}{ccc} \Omega & \hookrightarrow & \Lambda \\ \downarrow & & \vdots\,\exists! \\ \{\omega_{\mathcal{A}}\} & \hookrightarrow & \{\lambda_{\mathcal{A}}\} \end{array}$$

Now the set of identities E are represented in terms of a set of ordered pairs $\langle \omega, \omega' \rangle$, taken from (derived) operators with the same arity such that $\omega_{\mathcal{A}} = \omega'_{\mathcal{A}}$ hold.

In order to express a concrete algebraic structure in the framework, we would soon realize that it's required to dig into the rigorous construction of the relevant theory (e.g. Lawvere theory), by which a series of uncertainty will be vanished, such as *how the operators are exactly derived?* and *what kind of operations are permitted for the parameters, while the specified arity are taken into account? (ignore and/or copy parameters)*.

Aside from the detailed discussion, we can express a group G in the words of universal algebra, by operators $\Omega = \{\omega_0, \omega_1, \omega_2\}$, with the identities E corresponding to $ex = x = xe, xx^{-1} = e = x^{-1}x$ and $x(yz) = (xy)z$, where ω_0 is the assignment of identity $\langle\rangle \mapsto e$, ω_1 the assignment of the inverse $\langle x \rangle \mapsto x^{-1}$ and ω_2 the multiplication $\langle x, y \rangle \mapsto xy$.

4 A general method of constructing a colimit as the result of a left adjoint functor from Set

It is an immediate consequence of *adjoint functor theorem* that we have a **general method** to construct a colimit of certain type in a particular class of categories.

The applicable class of categories represented by C should suffice the following properties.

(a) C is locally small and complete small;
(b) It is given a continuous functor F to **Set**; (2)
(c) F suffices solution set condition (1).

Not exclusive yet particularly important instance of such classes is the *algebraic system* of fixed type, which forms a category of algebra $\mathbf{Alg}_\tau$ of given type τ. $\mathbf{Alg}_\tau$ suffices (2) along with the functor known as forgetful functor $U : \mathbf{Alg}_\tau \to \mathbf{Set}$.

If we are given such functor with these properties, *adjoint functor theorem* states that the continuous functor $F : C \to \mathbf{Set}$ admits a left adjoint, which preserves a colimit of **Set**, hence the image of the left adjoint is the constructed colimit in C.

Moreover, this construction procedure can be done in an algorithmic manner. Above lemma shows that, the solution set wInit(x ↓ F) of $x \in C$ can be identified with the union set of maps that span from x. This can be seen as follows:

To a given map $h : x \to Fa$, choosing a map $(u : x \to Fb) \in \mathrm{wInit}(x \downarrow F)$ through which h factors is equivalent to inducing a (part of) pullback map $u : x \to Fb$ in the diagrams of the form $Fs \to Fa \leftarrow Fb$.

This last description roughly speaks of algorithmic nature of a (co)limit preserving condition of procedure (i.e. functor), namely providing with a "basis" in the target category against an arbitrary object of source category.

Here we mean by "basis" a set of objects that yields a sort of irreducibleness (c.f. might be preferable to express as "atomicity" in a context).

4.1 Coproduct in Grp

In **Grp**, the free group construction F is the left adjoint to the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$, where a solution set of $s \in \mathbf{Set}$ is composed of every monomorphisms in the form of $s \to Ut$, having no factorization through another mono $s \to Ut'$. Hence s generates t freely as a group.

References

[1] P. M. Cohn. *Universal Algebra*. 1st ed. Mathematics and Its Applications. Springer, Apr. 1981. ISBN: 9027712549. URL: http://www.worldcat.org/isbn/9027712549.

[2] George Grätzer. *Universal Algebra*. Springer, 1968.

[3] Saunders MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262.