Note on the finality

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August 10, 2024

Definition. A functor $L : J' \to J$ is called **final** if for each $j \in J$, $(j \downarrow L)$ is non-empty and connected.

Exercise 1. [**p.295**, 1] The inclusion functor $\{j\} \hookrightarrow J$ of a discrete sub-category with single object $j \in J$ is final if and only if $j \in J$ is terminal.

Proof. If $j \in J$ is terminal, $(i \downarrow j)$ is singleton for each $i \in J$ and hence trivially is non-empty and connected. Conversely if $\{j\} \hookrightarrow J$ is final, $(i \downarrow j)$ is non-empty for each $i \in J$. For a pair of objects $i \rightrightarrows j$, there is a connected diagram from j to j that commutes with the arrows. The diagram coincides with identity since $\{j\}$ is discrete, hence $(i \downarrow j)$ is singleton. $\square$

Fact. Let $L : J' \to J$ be a final functor. Given a functor $F : J \to X$ that admits $x = \underrightarrow{\text{Colim}} FL$, then there exists $\underrightarrow{\text{Colim}} F$ that is isomorphic to x.

Proof. If a colimit cone $\mu : F \to \underrightarrow{\text{Colim}} F$ exists, then μ can be restricted onto J', namely:

$$\begin{array}{ccc} X^J & \xrightarrow{\mu} & X \\ {\scriptstyle L^*}\downarrow & \nearrow_{L^*\mu} & \\ X^{J'}. & & \end{array}$$

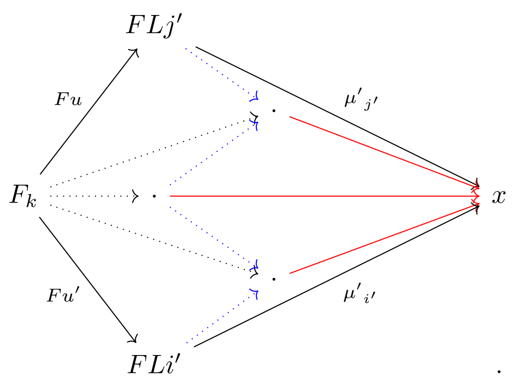
Therefore there is the canonical (unique) morphism h that commutes the following diagram:

$$\begin{array}{ccc} & & \underrightarrow{\text{Colim}} F \\ & \nearrow^{(L^*\mu)_{j'} = \mu_{Lj'}} & \uparrow{\scriptstyle h} \\ (FL)_{j'} & \longrightarrow & x. \end{array}$$

Given a colimit cone $\mu' : FL \to x$, μ is constructed by choosing $u : k \to Lj'$ for each $k \in J$ and composed accordingly:

$$Fk \xrightarrow{Fu} FLj' \xrightarrow{\mu'_{j'}} x, \qquad \mu_k = \mu'_{j'} \circ Fu.$$

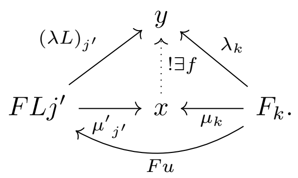
We see that μ_k is independent of the choice of j' and u since another choice of morphism $u' : k \to Li'$ admits a connected (zigzag) diagram of the form:



The dotted arrows arise from finality of L and hence all commute. All the triangles containing red arrows are commutative because μ' is a cone. These arguments are valid when any of blue arrows are reversed that conclude that μ_k is solely determined by k.

$\mu : F \dot{\to} x$ with the component μ_k for each $k \in J$ is a cocone from F since any morphism $k \to l$ in J induces a connected diagram between FLj'_k and FLj'_l that factors μ_k via μ_l, where FLj'_k denotes an arbitrary choice of object relaying Fk to x.

An another cocone $\lambda : F \dot{\to} y$ from F induces the cocone $\lambda L : FL \dot{\to} Ly = y$ from FL that factors through μ' (as in the left triangle of the following diagram). Because λ is a cocone, we have $\lambda_k = (\lambda L)_{j'} \circ Fu$ and hence $\lambda = f\mu$ is the unique factorization of the cocone λ.



□

Exercise 2. [**p.295**, 1] For any composable final functors $J \xrightarrow{L} J' \xrightarrow{L'} J''$, the composition $L'L$ is final.

Proof. For an arbitrary $i \in J''$, we can find $i \to L'j$ for some $j \in J'$, for which we can find $j \to Lk$ for some $k \in J$. By composition, we get $i \to L\jmath \to L'Lk$ in $(i \downarrow L'L)$. The connectivity is immediate from that of $(i \downarrow L')$. □

Exercise 3. [**p.295**, 1] Let $L : J' \to J$ be a full functor with J filtered. Then L is final if for any $k \in J$, $(k \downarrow L)$ is non-empty.

Proof. Because L is full, we only need to show that for any $(k \to Li)$, $(k \to Lj) \in (k \downarrow L)$, there exists a connected diagram between Li and Lj in J. Since J is filter, Li and Lj admit cocone $Li \to \cdot \leftarrow Lj$. □

References

[1] Saunders MacLane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262.