Sheafification as a free construction

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1 Sheafification

1.1 Introduction

In the world of algebra, we often encounter "free" constructions. Think of the free group or the polynomial ring—we take a set of generators and build the most general structure around them, adding only what's necessary. The process of sheafification, however, feels different. It's less like building from scratch and more like a sophisticated repair job. It takes a "broken" object called a presheaf, which understands local data but can't quite piece it together, and masterfully fixes its gluing mechanism to create a well-behaved sheaf.

This process might seem peculiar, but it's one of the most fundamental and powerful tools in modern geometry and topology. Without sheafification, much of the machinery of algebraic geometry would grind to a halt. It's the crucial step that allows us to construct cokernels, build resolutions, and ultimately connect the local properties of a space to its global invariants through the sheaf cohomology.

The idea of a sheaf wasn't born in a vacuum of pure abstraction. It has a compelling origin story, dating back to the brilliant work of French mathematician Jean Leray during his imprisonment in a POW camp in World War II. While studying the solutions to partial differential equations, he needed a way to organize and glue together local solutions into global ones. This concrete problem led him to invent the foundational concepts of sheaves and their cohomology, creating a language that would revolutionize mathematics for decades to come.

In this article, we'll demystify the construction of sheafification, exploring how it elegantly repairs the gluing axiom and provides the essential bridge from local data to global understanding.

We refer to the authoritative claim from a proposition in R. Hartshorne's "Algebraic Geometry".

Proposition 1. [**p.64**, 1]

Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, with the property that for any sheaf $\mathcal{Y}$, and any morphism $\phi : \mathcal{F} \to \mathcal{Y}$, there is the unique morphism $\psi : \mathcal{F}^+ \to \mathcal{Y}$ such that ϕ factors through θ.

In other words, the statement claims the existence of an universal arrow θ that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{Y} \\ \downarrow{\scriptstyle\theta} & \nearrow{\scriptstyle\psi} & \\ \mathcal{F}^+ & & \end{array}$$

The claim can be restated by θ being a free construction, or equivalently, θ is a left adjoint of the forgetful functor $G : \mathcal{F}^+ \to \mathcal{F}$; hence our strategy is to show that $\langle \mathcal{F}^+, \theta \rangle$ is the initial object of comma category $(\mathcal{F} \downarrow G)$.

First of all, for each open set $U \subseteq X$ define $\mathcal{F}^+$ as follows.

$$\mathcal{F}^+(U) := \left\{ s : U \to \bigcup_{p \in U} \mathcal{F}_p \,\middle|\, \begin{array}{l} (1)\, \forall p \in U, s(p) \in \mathcal{F}_p; \\ (2)\, \forall p \in U, \exists (p \in V \subseteq U), \exists t \in \mathcal{F}(V) \text{ s.t. } \forall q \in V, t_q = s(q). \end{array} \right\}$$

We denote the total space $\bigcup_{p \in U} \mathcal{F}_p$ over U by $|\mathcal{F}^+_U|$.

Proof. **Abelian group structure**. It follows immediately by setting $0_U = U \to 0$ and defining the addition pointwisely.

The restriction morphisms. Let us define the restriction morphism $\rho'^U_V : \mathcal{F}^+(U) \to \mathcal{F}^+(V)$ by $\rho'^U_V(s + s') = s|_V + s'|_V$, for $V \subseteq U$ and $s, s' \in \mathcal{F}^+(U)$, then we see that this is the morphism of abelian group with associative composition. Note that we can canonically regard a local section $s \in \mathcal{F}(U)$ as an element of $\mathcal{F}^+(U)$, considering a natural map $U \times \mathcal{F}(U) \to |\mathcal{F}^+_U|$ that sends $\langle p, s\rangle$ to the germ s_p, which is exactly the value of the element, i.e., function from U to $|\mathcal{F}^+_U|$, of $|\mathcal{F}^+_U|^U$ by exponential law. With this observation, ρ' is compatible with the morphism θ of presheaves, namely, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\rho^U_V} & \mathcal{F}(V) \\ \downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_V} \\ \mathcal{F}^+(U) & \xrightarrow{\rho'^U_V} & \mathcal{F}^+(V). \end{array}$$

The locality. Given an open cover $U = \bigcup_i U_i$ and $s, s' \in \mathcal{F}^+(U)$, assume $s|_{U_i} = s'|_{U_i}$ for all i. Then $\forall p \in U, \exists i$ such that $s(p) = s'(p)$ over $U_i \ni p$. By definition of the stalk, the restriction to any smaller neighbourhoods of p doesn't affect the germ at p and since $p \in U$ is arbitrary, $s = s'$.

The glueing of local sections. For an arbitrary open cover $U = \bigcup_i U_i$ of open set $U \subseteq X$, assume that any local sections $s_i \in \mathcal{F}^+(U_i)$ and $s_j \in \mathcal{F}^+(U_j)$ suffice $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any i, j such that $U_i \cap U_j \neq \varnothing$. Then, by (2) there exists an open set $V \subset U_i \cap U_j$ and $t \in \mathcal{F}(V)$ such that $t|_V = s_i|_V = s_j|_V$ (More rigorously, (2) only guarantees the existence of a local section t that coincides with s_i and s_j in each stalk $\mathcal{F}_p$ in some open set $V' \subset U_i \cap U_j$. By definition of germ, equivalent class of pair $\langle s, V\rangle$, defining the same element $s_{i,p} = s_{j,p}$ in $\mathcal{F}_p$ admit a smaller neighbourhood $V \subset V'$ where $s_i = s_j$ holds. By uniqueness, we can set $t = s_i$).

Therefore it is well-defined to set $s(p) := s_i(p)$ for any $p \in U$ and $U_i \ni p$.

The universality. From the morphism ϕ of presheaf, observe that there is an uniquely induced morphism $\phi_p : \mathcal{F}_p \to \mathcal{Y}_p$ between each stalks, shown in the diagram.

$$\begin{array}{ccccc} & & \mu^U_V & & \\ \mathcal{Y}(U) & \longrightarrow & \mathcal{Y}_p & \longleftarrow & \mathcal{Y}(V) \\ {\scriptstyle \phi_U}\uparrow & & {\scriptstyle \phi_p}\uparrow & & {\scriptstyle \phi_V}\uparrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}_p & \longleftarrow & \mathcal{F}(V). \\ & & \rho^U_V & & \end{array}$$

Furthermore, we have the induced map $\theta_p : \mathcal{F}_p \to \mathcal{F}^+_p$ of stalks that sends each s_p to a map $\{p\} \to s_p$, which is an isomorphism as in the diagram below.

$$\begin{array}{ccccc} & & \rho'^U_V & & \\ \mathcal{F}^+(U) & \longrightarrow & \mathcal{F}^+_p & \longleftarrow & \mathcal{F}^+(V) \\ {\scriptstyle \theta_U}\uparrow & & {\scriptstyle \theta_p}\uparrow & & {\scriptstyle \theta_V}\uparrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}_p & \longleftarrow & \mathcal{F}(V). \\ & & \rho^U_V & & \end{array}$$

Composing there maps, we have a map $\psi_p = \phi_p \circ \theta_p^{-1} : \mathcal{F}^+_p \to \mathcal{Y}_p$ of stalks to deduce the unique map ψ_U of local sections that is compatible in a way that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}^+(U) & \xrightarrow{\psi_U} & \mathcal{Y}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}^+_p & \xrightarrow{\psi_p} & \mathcal{Y}_p \end{array}$$

The construction of ψ_U is similar as in the previous process: for each section $s \in \mathcal{F}^+(U)$, consider a continuous assignment of $U \ni p \mapsto s_{V_p} \in \bigcup_p \mathcal{F}(V_p)$, which is a composition of the evaluation map $e_p(s) = s_p$ followed by a choice of germ representative $c(s_p) = s_{V_p}$ at some neighbourhood V_p, with the property that $s_p = s_{V_p}|_p$. With this assignment, we can identify the section s as the collection of pairs $\{\langle s_{V_p}, V_p \rangle\}_{p \in U}$, for which we simply denote $\{s_{V_p}\}_{p \in U}$.

We claim that $\{\phi_{V_p}(s_{V_p})\}_{p \in U}$ gives rise to the unique element t of local section $\mathcal{Y}(U)$, and we define $\psi(s) = t$.

When we are given such collection $\{\phi_{V_p}(s_{V_p})\}_{p \in U}$, it is clear that the recovery of t is well defined by $t|_{V_p} = \phi_{V_p}(s_{V_p})$ and that the sheaf properties of $\mathcal{Y}$ admits the uniquely glued local section $t \in \mathcal{Y}(U)$. If one chose two representatives s_{V_p} and s_{W_p} of the germ s_p, by definition of germ equivalence, we have a neighbourhood $Z_p \subset V_p \cap W_p$ on that s_{V_p} and s_{W_p} define the same section. $\square$

References

[1] Robin Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer, 1977. URL: http://www.worldcat.org/oclc/2798099.