The Fractions

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January 6, 2026

Fractions are deceptively difficult. I have seen many parents agonize over how to explain concepts like 1/4 or 2/5 to an eight-year-old without simply resorting to rote formal operations. Paradoxically, I only fully realized the complexity of basic fractions by taking a detour into deep theory. While reading Weibel's *An Introduction to Homological Algebra*, I found a fascinating exposition on the categorification of fractions—known as localization.

A quick technical note. For the sake of rigor, we assume all categories discussed here belong to a Grothendieck universe and that the multiplicative system is locally small on the left. However, these set-theoretic details are not the focus of the article.

1 Multiplicative System

Definition 1. *A* multiplicative system *S in a category C is a collection of morphisms that satisfies the following three self-dual axioms:*

1. S is closed under composition and contains all identity morphisms for all objects in C;
2. **(Ore condition)** For each pair $g \in \mathrm{Mor}(C)$, $t \in S$ with

   $$X \xrightarrow{g} Y \xleftarrow{t} Z,$$

   there exists a weak pullback $f \in \mathrm{Mor}(C)$, $s \in S$ such that $gs = tf$ in C. The dual statement also holds (existence of a weak pushout).
3. **(Cancellation)** For each pair of parallel morphisms $f, g : X \to Y$ in C, the following are equivalent:
   (a) there exists a weak coequalizer of f, g in S, namely $sf = sg$ for some $s \in S$ with $\mathrm{dom}(s) = Y$;
   (b) there exists a weak equalizer of f, g in S, namely $ft = gt$ for some $t \in S$ with $\mathrm{codom}(t) = X$.

1.1 The Spirit of the Ore Condition: Right (Left) Permutability

Origin of the Name

The term comes from the Norwegian mathematician Øystein Ore. In 1931, he studied the problem of embedding a non-commutative ring into a division ring. He discovered that one cannot always form a field of fractions for a non-commutative ring; this is possible only if the ring satisfies the Ore condition.

Intuition

In the localized category $C[S^{-1}]$, we want to compose a "fraction" fs^{-1} with a morphism g:

$$(X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y) \circ (Y \overset{g}{\rightarrow} Z).$$

This composition is $g \circ f \circ s^{-1}$, which is well-defined.

However, composing two fractions,

$$(fs^{-1}) \circ (gt^{-1}),$$

formally gives $f \circ s^{-1} \circ g \circ t^{-1}$. To rewrite this as a single fraction, we must move s^{-1} past g. That is, we seek g' and s' such that

$$s^{-1} \circ g = g' \circ (s')^{-1},$$

or equivalently,

$$g \circ s' = s \circ g'.$$

This is precisely the Ore condition. It ensures that any zigzag can be reduced to a single roof (a span $X \overset{s}{\leftarrow} Z \overset{f}{\rightarrow} Y$).

1.2 The Spirit of Cancellation: Zero Divisors

In ring theory,

$$\frac{a}{s} = \frac{b}{s} \iff t(a-b) = 0 \text{ for some } t \in S,$$

which implies $ta = tb$.

Categorically, this corresponds to the existence and cancellability of morphisms in S. The equivalence of conditions (3.a) and (3.b) ensures that it does not matter on which side one multiplies, enforcing that elements of S behave like isomorphisms.

With these axioms, every morphism in $C[S^{-1}]$ is representable as a fraction fs^{-1}.

2 Localization

Definition 2. *Let S be a collection of morphisms in a category C. A* localization *$C[S^{-1}]$ is a category together with a universal functor $q : C \to C[S^{-1}]$ such that any functor $F : C \to D$ sending every $s \in S$ to an isomorphism factors uniquely through q.*

This definition ensures uniqueness of $C[S^{-1}]$ up to equivalence, and that $q(s)$ is an isomorphism for all $s \in S$.

However, the definition alone does not provide a concrete description of morphisms in $C[S^{-1}]$. This is the problem of constructability.

3 Construction of Localization

Calculus of Fractions

There are two primary constructions: the general zig-zag construction and the calculus of fractions.[1]

[1] Weibel follows the calculus of fractions approach, defining morphisms via equivalence classes of roofs. [1]

Definition 3. *Given a multiplicative system S in a category C, a* (left) fraction *is a diagram*

$$fs^{-1} : X \overset{s}{\leftarrow} X_1 \overset{f}{\rightarrow} Y,$$

where $s \in S$ and $f \in \mathrm{Mor}(C)$.

Let $F(S)$ denote the collection of all such fractions. An equivalence relation $\sim$ on $F(S)$ yields

$$C[S^{-1}] \cong F(S)/\sim.$$

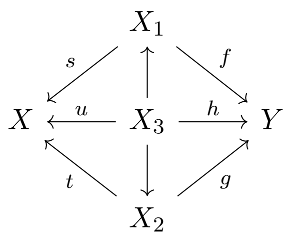
Definition 4. *Two fractions*

$$X \overset{s}{\leftarrow} X_1 \overset{f}{\rightarrow} Y \quad \textit{and} \quad X \overset{t}{\leftarrow} X_2 \overset{g}{\rightarrow} Y$$

are equivalent if there exists a third fraction

$$X \overset{u}{\leftarrow} X_3 \overset{h}{\rightarrow} Y$$

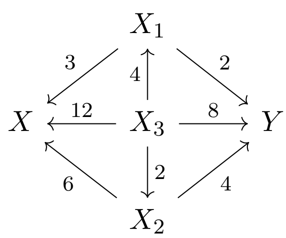
such that the following diagram commutes:



This is known as a *common roof* or *common span*. It plays the role of a common denominator in ordinary fractions.

In commutative algebra, $\frac{a}{s} = \frac{b}{t}$ if and only if $at = bs$. In a general category, domains differ and morphisms do not commute, so we require a common refinement.

An illustrative numerical analogy is:



Finally, defining equivalence via a single morphism $X_1 \to X_2$ would destroy symmetry. The common roof construction ensures symmetry and transitivity, relying crucially on the Ore condition.

References

[1] Charles A. Weibel. *An Introduction to Homological Algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1994. ISBN: 978-0-521-55987-4.