# Meaning of $(n, r)$-Category

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May 22, 2026

## A useful notation with unstable boundaries

The notation $(n, r)$-category is meant to classify higher categories by two parameters, where $n$ denotes the cutoff dimension for non-trivial higher morphisms (often called cells) and $r$ denotes the cutoff dimension for non-invertible higher morphisms.

The notation is useful because it compresses several familiar cases into one scheme: sets, preorders, groupoids, ordinary categories, bicategories with invertible 2-cells, and so on. But it is also treacherous, because the words "nontrivial," "equivalent," and "invertible" do not have model-independent meanings in higher category theory. They are interpreted differently in strict globular categories, weak $n$-categories, quasicategories, complete Segal spaces, and enrichment-based definitions.

The result is that $(n, r)$ is better understood not as a single definition, but as a notational regime: a compact way of saying what kind of higher categorical truncation and invertibility one has in mind, provided the ambient model has already been specified.

## The naïve reading

The first reading one usually learns is[1]:

> An $n$-category has $k$-morphisms for $0 \leq k \leq n$, and no higher morphisms.

Thus a 0-category is a set, a 1-category is an ordinary category, and a 2-category is a 1-category with additional 2-morphisms and coherence

[1] Some older sources instead formulate this in terms of the existence of morphisms.

data through dimension 2. With this strict dimensional reading, a strict $(n+1)$-category may be defined inductively as a category enriched in strict $n$-categories.

But the notation $(n, r)$ was not designed merely to say how many layers of cells exist; it also tracks invertibility, as the examples display: $(1, 1)$ corresponds to ordinary categories, $(1, 0)$ to groupoids, and $(\infty, 1)$ to $\infty$-categories with all morphisms above dimension 1 invertible.

Already here, the phrase "$k$-morphisms are trivial" is ambiguous, and indeed it is model dependent.

## "Trivial" does not always mean "absent"

The main source of confusion is that "trivial above dimension $n$" has several possible meanings.

In a strict model, it may mean that all $k$-morphisms for $k > n$ are identities. In a simplicial or homotopical model, it may instead mean that parallel higher cells are equivalent in a prescribed sense, or that the relevant mapping spaces are suitably truncated.

For example, Lurie's *Higher Topos Theory* defines an $n$-category inside simplicial sets by imposing uniqueness conditions on simplices above dimension $n$: if $f, f' : \Delta^m \to \mathcal{C}$ have the same restriction to $\partial\Delta^m$, then $f = f'$ for $m > n$. In dimension $n$, it requires relatively homotopic simplices to be equal. This is not the same as merely saying that no higher formal symbols exist [Lurie, Definition 2.3.4.1].

Bergner–Rezk compare several models for $(\infty, n)$-categories by explicit Quillen equivalences [Bergner–Rezk I, Bergner–Rezk II]. Separately, Lurie proves that an $\infty$-category is categorically equivalent to an $n$-category if and only if each of its mapping spaces is $(n-1)$-truncated [Lurie, Proposition 2.3.4.18]. In the quasicategorical setting, this is equivalent to the right lifting property with respect to $\partial\Delta^m \to \Delta^m$ for every $m \geq n+2$ [Campbell–Lanari, Proposition 3.12]. Campbell–Lanari also construct the corresponding model structure as the Bousfield localization of the Joyal model structure at $\partial\Delta^{n+2} \to \Delta^{n+2}$ [Campbell–Lanari, Theorem 3.28].

One of the most revealing examples is $(0, 1)$. One might think $n = 0$ should mean that there are no meaningful 1-morphisms, forcing us to see a $(0, 1)$-category merely as a set.

In Lurie's on-the-nose simplicial-set definition, however, a 0-category is literally the nerve of a partially ordered set [Lurie, Example 2.3.4.3]. If arrows exist in both directions between objects $x$ and $y$, the uniqueness condition forces the composites to be identities, and Lurie's equality condition in dimension 0 then forces $x = y$.

The equivalence-invariant characterization gives a different presentation. Requiring mapping spaces to be $(-1)$-truncated allows an ordinary thin category, hence a preorder, with distinct but isomorphic objects; it is categorically equivalent to the nerve of its posetal skeleton [Lurie, Proposition 2.3.4.18] [Campbell–Lanari, Proposition 3.10]. Thus preorder versus poset is not uniformly an external skeletal-replacement issue. The distinction depends on whether truncation is imposed on the nose or only up to categorical equivalence.[2]

In either presentation, 1-morphisms need not be absent. Instead, their multiplicity is suppressed: between two objects there is at most one arrow on the nose in a thin ordinary category, or at most one up to the relevant equivalence in a homotopical model. The parameter $r = 1$ permits such arrows to be non-invertible.

## Invertibility is also relative

The other unstable word is "invertible," or, in a weak setting, "equivalent."

In ordinary category theory, a morphism $f : a \to b$ is invertible if there exists $g : b \to a$ such that $gf = \mathrm{id}_a$ and $fg = \mathrm{id}_b$. No higher cell is needed to witness these equalities.

In a weak higher category, the equations may instead be replaced by equivalences $gf \simeq \mathrm{id}_a$ and $fg \simeq \mathrm{id}_b$, together with the coherence data required by the chosen model.

This difference matters, since it changes what counts as invertible.

## Why the notation takes this form

One structural reason is coherence.

In strict low-dimensional category theory, one can say "there are no higher morphisms." But weak higher categories require coherence cells. In sufficiently weak theories, one cannot simply delete higher cells without damaging the structure being defined, which explains the shift from "no cells above dimension $n$" to "cells above dimension $n$ are trivial, degenerate, or uniquely determined" in the chosen regime.

[2] Lurie explicitly notes that his on-the-nose definition is not invariant under categorical equivalence [Lurie, Remark 2.3.4.6]. A preorder can therefore present the same categorical equivalence type as a poset without itself satisfying that strict definition of a 0-category.

The literature reflects this. There is no single universally accepted model of weak finite $n$-categories; Leinster's survey documents a substantial range of competing definitions [Leinster]. This is a statement about the finite weak case, not about the comparison theory of $(\infty, n)$-categories.

For $(\infty, n)$-categories, the situation is better. There are comparison and unicity results for major proposed models: Bergner–Rezk establish Quillen equivalences among several model structures for $(\infty, n)$-categories, while Barwick–Schommer-Pries give an axiomatic unicity theorem for the homotopy theory of $(\infty, n)$-categories. But this does not automatically settle all questions about finite weak $n$-categories, nor about the expressive completeness of the two-parameter notation $(n, r)$ [Bergner–Rezk I, Bergner–Rezk II, Barwick–Schommer-Pries].

For convenience, I internally read the dimension $n$ as a truncation, thinness, or nontriviality parameter, while the dimension $r$ is read as the invertibility parameter, both depending on the prescribed equivalence or ambient structure.

## The current landscape

For present purposes, I find it useful to distinguish three levels of usage.

First, in informal exposition, $(n, r)$ is used as a mnemonic. It tells the reader approximately where nontriviality and non-invertibility stop. This is common in web/blog-style explanation and casual higher-categorical discussion.

Second, in serious technical work, authors usually specify a model: quasicategories, complete Segal spaces, marked or scaled simplicial sets, or enriched $\infty$-categories. Once the model is fixed, "trivial," "equivalent," and "invertible" acquire precise meanings.

Third, in recent foundational work, there is renewed interest in organizing the entire family $\mathrm{Cat}_{(n,r)}$ systematically. Goldthorpe sets $\mathrm{Cat}_{(n,0)} = \mathrm{Grpd}_n$ and defines the remaining cases recursively by enrichment; this is a serious structured use of $(n, r)$-notation, but it is not primarily a meta-analysis of the notation itself [Goldthorpe].

There is also work on the tower $\cdots \mathrm{Cat}_{(\infty,r)} \subseteq \mathrm{Cat}_{(\infty,r+1)} \subseteq \cdots$ with adjoints that either invert or discard certain higher morphisms. Goldthorpe distinguishes $\pi_{\leq r}$, which formally inverts $(r+1)$-morphisms, from $\kappa_{\leq r}$, which keeps the maximal sub-$(\infty, r)$-category by discarding noninvertible $(r + 1)$-morphisms [Goldthorpe].

## The problem of representativeness

Does $(n, r)$ represent all meaningful higher-categorical distinctions? The answer is no. It captures two important dimensions:

- where higher structure becomes trivial or truncated;
- where higher morphisms become invertible.

But it does not record:

- whether equality is strict or weak;
- whether equivalent objects are identified;
- whether higher equivalences are inductive, coinductive, saturated, or reversible;
- whether the model is globular, simplicial, opetopic, cubical, Segal-type, or type-theoretic;
- whether one works internally to spaces, sets, simplicial sets, model categories, or $\infty$-topoi;
- whether truncation means absence, identity-only, propositionality, contractibility, or a lifting property.

Thus $(n, r)$ is expressive but incomplete. It is a coordinate label, not a full specification.

## A convention for a better use

The $(0, 1)$-case is especially prone to confusion when it is expressed without qualification as

$$(0, 1)\text{-category} = \text{partially ordered set},$$

or something similar.

The same issue recurs at higher dimensions. A weakly truncated object need not satisfy an on-the-nose truncation condition. An equivalence class of structures is not the same as a chosen skeletal representative. The $(n, r)$-notation alone does not say which level of identification is intended.

The notation remains valuable. But its value is heuristic and classificatory before it is definitional. A mathematically precise use of $(n, r)$ should always say what model, what equivalence, and what truncation convention is being used.

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